HAMILTONIAN DYNAMICS OF A PARTICLE SUBJECT TO TIME-VARYING GYROSCOPIC AND POTENTIAL FORCES

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Summary We examine the planar dynamics of a particle under the simultaneous influence of a gyroscopic force and a force given by the gradient of a spatial potential. We show that this system possesses a noncanonical Hamiltonian structure in general, and that a reduced structure is present when the gyroscopic force corresponds to a simple model for aerodynamic lift. We also show that when the spatial potential varies sinusoidally with time, Poincaré sections of the reduced phase space can exhibit fractal structure.

INTRODUCTION

Consider a particle with mass m that moves in the (x, y) plane subject to both a gyroscopic force with magnitude F and a force given by the spatial gradient of a potential function $\phi(x, y)$. In terms of the linear momenta $p_x = m\dot{x}$ and $p_y = m\dot{y}$, the motion of such a particle is governed by the system of equations

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \frac{F}{\sqrt{p_x^2 + p_y^2}} \begin{bmatrix} -p_y \\ p_x \end{bmatrix} + \begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{bmatrix}. \tag{1}$$

We may regard x and y as coordinates on the particle's configuration manifold $Q = \mathbb{R}^2$ and regard p_x and p_y as corresponding coordinates induced on each cotangent space T_q^*Q.

HAMILTONIAN STRUCTURE

Proposition 1. *The system comprising* (1) *and the definitions* $p_x = m\dot{x}$ *and* $p_y = m\dot{y}$ *is Hamiltonian with respect to the Hamiltonian function* $H : T^*Q \to \mathbb{R} : (x, y, p_x, p_y) \mapsto \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \phi(x, y)$ *and the symplectic form*

$$\Omega = dx \wedge dp_x + dy \wedge dp_y + \frac{Fm}{\sqrt{p_x^2 + p_y^2}} dx \wedge dy \tag{2}$$

on T^*Q.

Proof. This follows from direct calculation. □

Note that Proposition 1 requires no assumptions about the dependence of F on x, y, p_x, or p_y and no assumptions about the direct dependence of F or ϕ on time. In [1], the author presented a preliminary analysis of the dynamics of this system in the case in which the gyroscopic force F depends quadratically on the translational speed of the particle and linearly on a time-varying control parameter λ. In this case, the system represents a model for the dynamics of a free circular cylinder moving through a planar fluid, spinning about its center at a time-varying speed to generate a time-varying lift. It was demonstrated in [2] that the lift on a spinning cylinder translating steadily through a planar fluid depends quadratically on the cylinder's translational speed and linearly on its spinning speed; a coarse validation of this model for unsteady translation was included in [1], based on experiments with an aquatic vehicle steered with a submerged spinning rod.
A lift-like dependence on translational speed indicates that the gyroscopic force in (1) takes the form $F = \lambda\left(p_x^2 + p_y^2\right)$. With this substitution, the symplectic form (2) becomes

$$\Omega = dx \wedge dp_x + dy \wedge dp_y + \lambda m\sqrt{p_x^2 + p_y^2}\, dx \wedge dy \tag{3}$$

and the equations of motion (1) become

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \lambda\sqrt{p_x^2 + p_y^2} \begin{bmatrix} -p_y \\ p_x \end{bmatrix} + \begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{bmatrix}. \tag{4}$$

It's intuitively clear — and readily verified by inspection — that if $\phi = 0$ identically, then the magnitude $\sqrt{p_x^2 + p_y^2}$ of the particle's forward momentum will be conserved, since the force it experiences is always perpendicular to its velocity. Under this assumption and the assumption of constant λ, the equations of motion above were obtained via Hamiltonian reduction in [3], framed not as the equations of motion for a steadily spinning cylinder but as those for a cylinder supporting a constant fluid circulation around its boundary.

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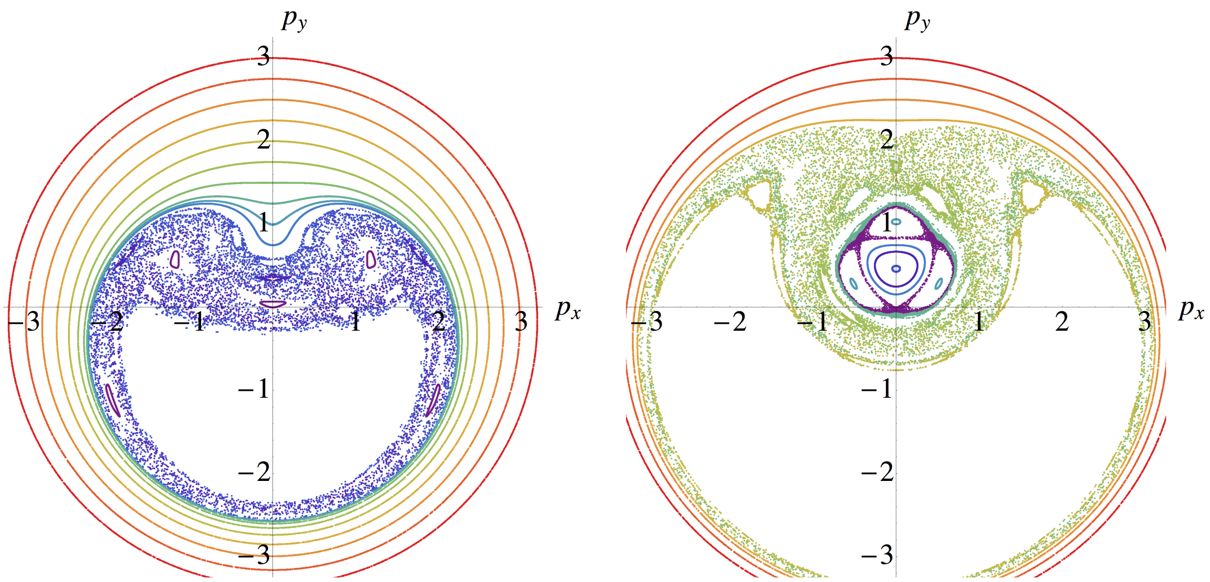

Figure 1. Stroboscopic images of solutions to (4) with $\lambda = 1$ and $\phi = -y \sin \nu t$. The plot on the left corresponds to $\nu = 1.4$ and the plot on the right to $\nu = 2.2$. In each plot, a single color is assigned to the solution obtained from one of thirteen initial conditions spaced equally between $(p_x, p_y) = (0, 0)$ and $(p_x, p_y) = (0, 3)$. Each solution is sampled at time $t = 2k\pi/\nu$ for $\nu = 1, \ldots, 5000$. High-resolution versions of these images and others corresponding to different frequencies ν are available at `http://kellyfish.net/ictam12/`.

Proposition 2. *If $\phi = Ax + By$, then the system* (4) *is Hamiltonian with respect to the Hamiltonian function $h : \mathbb{R}^2 \to \mathbb{R} : (p_x, p_y) \mapsto -\frac{1}{3}\lambda \left(p_x^2 + p_y^2\right)^{3/2} + Ap_y - Bp_x$ and the symplectic form $\omega = dp_x \wedge dp_y$ on $\mathbb{R}^2$.*

Proof. This follows from direct calculation. □

Note that the coefficients A and B may depend directly on time; Proposition 2 requires only that they be independent of x and y. The Hamiltonian structure in Proposition 2 represents a reduction of that in Proposition 1 in the sense that the phase space of the latter is lower dimensional than that of the former. The dynamics on the larger phase space can be reconstructed from the dynamics on the smaller phase space — in other words, $x(t)$ and $y(t)$ can be obtained from $p_x(t)$ and $p_y(t)$ after the latter are determined by (4) — via separate integration with respect to time.

DYNAMICS

If λ is constant and $\partial\phi/\partial x = \partial\phi/\partial y = 0$, solutions to (4) correspond to circular orbits around the origin in the (p_x, p_y) plane with period $2\pi/|\lambda|\sqrt{p_x(0)^2 + p_y(0)^2}$, executed clockwise if λ is positive and counterclockwise is λ is negative. If λ or ϕ varies periodically with time, the system's orbits become much more complex, even if the spatial gradient of ϕ is constant. The case in which λ varies sinusoidally with time was considered briefly in [1]; we examine the complementary case now in which λ is constant and ϕ varies sinusoidally.

Figure 1 depicts two Poincaré sections reflecting the dynamics of (4) induced by sinusoidal oscillations in a potential function ϕ satisfying the conditions of Proposition 2. The apparently solid curves surrounding each plot comprise collections of discrete points along distinct trajectories. Certain initial conditions lead to trajectories which, when sampled stroboscopically, describe fractal sets rather than curves, suggesting chaotic behavior. We assess the fractal nature of such sets by computing the correlation dimension of each, following the procedure outlined in [4]. The trajectory corresponding to the initial condition $(p_x, p_y) = (0, 1/2)$ in the left-hand plot is represented, for instance, by a diffuse collection of blue points with approximate correlation dimension 1.8. The trajectory corresponding to the initial condition $(p_x, p_y) = (0, 2)$ in the right-hand plot is represented by a diffuse collection of green points with approximate correlation dimension 1.2.

References

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