A VIBRATION ANALYSIS OF QUARTZ CRYSTAL RESONATORS WITH NONLINEAR MINDLIN PLATE THEORY

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Summary We have derived the nonlinear Mindlin plate theory to consider the nonlinear properties of materials for the study of vibration properties of quartz crystal resonators. We also included the strong electrical field for frequency effects and found that larger frequency shift is resulted due its presence. The separated calculations of frequency shifts under different loadings are important in the analysis of nonlinear behaviour of quartz crystal resonators and comparison can be made with the current results from actual measurements. The solutions we have obtained are also consistent with general trends presented before.

INTRODUCTION

The Mindlin plate theory has been widely known with surprisingly focused applications in the analysis of quartz crystal resonators for frequency control and detection which is not well known to the outside community. The fundamental reason for such usage is due to the effect that the Mindlin plate theory can consider the thickness-shear vibrations, which are also rotations in the deformation and have higher frequencies as functioning modes of thickness type resonators. The linear theory has been widely used in the analysis and design of quartz crystal resonators with useful results like the vibration frequency, mode shapes, and optimal structural parameters as part of the design procedure based on theoretical analysis in combination with experimental procedure [1-2]. The plate theory has also been extended to consider complication factors of quartz crystal resonators such as electrodes and bias fields in operations. The challenges of resonator design and technology have been on maintaining the superb performance of quartz crystal resonators with excellent quality factor and stability of working frequency while minimizing frequency shifts under all operating conditions. To this goal, as theoretical analysis and empirical observations indicate, the linear theory is no longer enough and nonlinearity of material, structure, and bias fields has to be considered. Such research has been done by many investigators and results have been obtained as part of the process to improve the design process. It is generally expected that such studies have been useful in providing design tips and eventual improvement of performance in many mission critical applications.

NONLINEAR MINDLIN PLATE THEORY

The systematic derivation of the Mindlin plate equation is best presented as the power series expansion of displacements in the thickness coordination and subsequent reduction of the three-dimensional equations of elasticity or piezoelectricity to the two-dimensional set through a standard procedure of variational formulation. Such a procedure starts with

$$u_j(x_1, x_2, x_3, t) = \sum_{n=0}^{\infty} u_j^{(n)}(x_1, x_3, t) x_2^n,\ j = 1,2,3, \tag{1}$$

where $u_j, x_i(i = 1,2,3), t,$ and $u_j^{(n)}$ are displacements, coordinates, time, and higher-order displacements, respectively. It should be emphasized that x_2 is the thickness coordinate and it is eliminated from the integration. With (1), we have strains as

$$S_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k} + u_{m,k}u_{m,l}\right), k, l, m = 1,2,3. \tag{2}$$

And the higher-order strain components in the Green-Lagrangian form are

$$\begin{aligned} S_{kl}^{(n)} = &\frac{1}{2}\left[u_{k,l}^{(n)} + u_{l,k}^{(n)} + (n+1)\left(\delta_{2l}u_k^{(n+1)} + \delta_{2k}u_l^{(n+1)}\right)\right] \\ &+ \frac{1}{2}\left\{\sum_{s=0}^{n}\left[u_{m,l}^{(s)} + \delta_{2l}(s+1)u_m^{(s+1)}\right]\cdot\left[u_{m,k}^{(n-s)} + \delta_{2k}(n-s+1)u_m^{(n-s+1)}\right]\right\}. \end{aligned} \tag{3}$$

It should be pointed out that we have the nonlinear forms of strains, or the kinematic nonlinearity, now. By utilizing the variational nonlinear equation of motion

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$$\int_V \left[\left(T_{ij}+T_{ik}u_{j,k}\right)_{,i}-\rho\ddot{u}_j\right]\delta u_j dV=0, \tag{4}$$

the two-dimensional equations of motion are

$$\left[T_{ij}^{(n)}+\sum_{m=0}^{\infty}T_{ik}^{(m+n)}\bar{S}_{jk}^{(m)}\right]_{,i}-nT_{2j}^{(n-1)}+F_j^{(n)}+\bar{F}_k^{(n)}-nT_{2k}^{(m+n-1)}\sum_{m=0}^{\infty}\bar{S}_{jk}^{(m)}-\rho\sum_{m=0}^{\infty}B_{mn}\ddot{u}_j^{(m)}=0. \tag{5}$$

And the constitutive relations in this case will be

$$T_{ij}=c_{ijkl}S_{kl}+\frac{1}{2}c_{ijklmn}S_{kl}S_{mn}=\sum_{n=0}^{\infty}\left[c_{ijkl}S_{kl}^{(n)}+\frac{1}{2}c_{ijklmn}S_{klmn}^{(n)}\right]x_2^n, \tag{6}$$

where the material nonlinearity is considered with the presence of higher-order elastic constants c_{ijklmn}.

APPROXIMATIONS AND SOLUTIONS

The nonlinear Mindlin plate equations above are much more complicated because of the presence of coupling terms due to material and kinematic nonlinearities. As a result, even single thickness-shear vibrations which can give the vibration frequency and mode shape of cannot be solved analytically. Clearly, the essential vibrations involving the thickness-shear and flexural modes, which are strongly coupled in the linear case, have to be studied with simplified equations by dropping all other coupled modes. Then, the partial differential equations have to be converted to ordinary differential equations for possible solutions of frequency-response relations [3]. For the single thickness-shear mode, we utilized both perturbation and the homotopy analysis method (HAM), a newer method for strong nonlinear equations, for the frequency-amplitude relations [4]. Fig.1 showed that the results from the analytical solutions and our finite element implementation are close to each other, and they are also consistent with earlier studies. Furthermore, we extended the nonlinear equations with the presence of the electrical field in terms of electrical potential and accompanying charge equations to study the electrical effect. Fig.2 showed that the frequency shifts due to driving voltages (current or power) are much larger, demonstrating strong effects of electrical field.

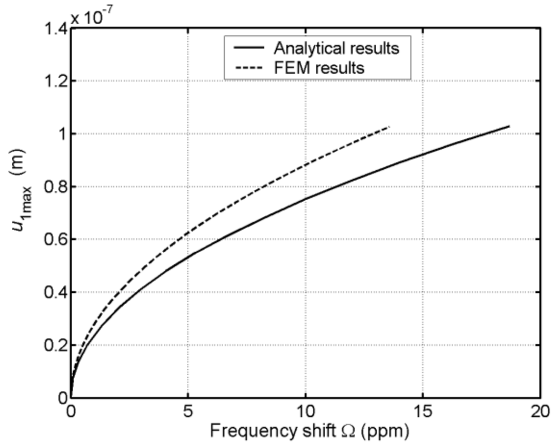

Fig.1. Nonlinear amplitude-frequency relations for free vibrations.

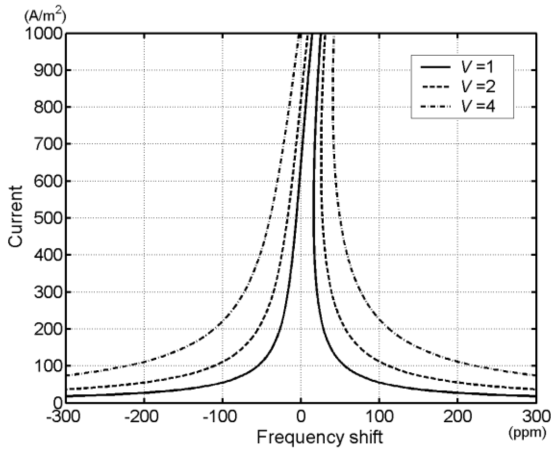

Fig.2. Frequency-response curve with different voltage ($\alpha=0.1$).

CONCLUSIONS

A set of successive higher-order nonlinear Mindlin plate equations with the consideration of material and kinematic nonlinearities and strong electrical fields has been established and approximate results have been obtained with analytical methods. These solutions will be important in the analysis and design of quartz crystal resonators for better performance prediction under different loading and bias fields.

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