COMPUTING BIFURCATIONS OF DISSIPATIVE SYSTEMS VIA POD-BASED ROMs

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Summary This paper illustrates how reduced order models (ROMs) based on proper orthogonal decomposition (POD) can be used to efficiently construct bifurcation diagrams involving dissipative systems. The proposed method relies on the observation that POD manifolds resulting from snapshots calculated for specific values of the parameters of the problem also contain the attractors for other parameters values. Application to a fairly complex bifurcation diagram for the Ginzburg-Landau equation is considered and discussed.

INTRODUCTION

Construction of bifurcation diagrams is a relevant issue in many scientific problems. On the other hand, bifurcations are common in fluid flows of industrial interest, thus they will be unavoidably important in industrial design and certification. Computation of general bifurcations, however, may require huge computational resources when the considered physical system is modeled by PDEs. In fact, it is usually necessary to solve the underlying evolution problem many times during a fairly large time span (thus discarding transient behaviors) in order to calculate and continue the involved steady, periodic, quasi-periodic, and chaotic attractors. But if the governing equations are dissipative, POD may provide a low dimensional linear manifold containing the large time behavior of the system. In this context, POD-based ROMs have been successfully used by many authors [1, 5, 7]. In bifurcation problems, the appropriate snapshots are usually selected "ad hoc", close to a specific bifurcation point, in order to approximate the bifurcation diagram in a neighborhood [3, 4].
Against this background, the object of this work is showing some simple ideas that can be exploited in order to get flexible, accurate, and fast ROMs approximations in bifurcation problems involving dissipative systems.

POD-BASED ROMs IN BIFURCATION PROBLEMS

Let us consider a dissipative system of semilinear parabolic equations of type

$$\partial_t \boldsymbol{q} = \boldsymbol{\mathcal{L}} \boldsymbol{q} + \boldsymbol{f}(\boldsymbol{q}, \mu), \tag{1}$$

where $\boldsymbol{q}$ is a state vector in $\boldsymbol{E}$, assumed to be a Hilbert space, and μ is a bifurcation parameter. In addition, $\boldsymbol{\mathcal{L}}$ is a (generally unbounded) linear operator such that $\boldsymbol{\mathcal{L}}^{-1}$ is well defined and compact, and $\boldsymbol{f}$ is a nonlinear operator such that $\boldsymbol{q} \to \boldsymbol{\mathcal{L}}^{-1}\boldsymbol{f}(\boldsymbol{q}, \mu)$ is compact. These assumptions include, e.g., the reaction-diffusion and the micro-fluidic systems.
Let us then suppose to have calculated (by a time-dependent numerical solver) a representative set of snapshots, whose POD provides a good manifold spanned by an ordered set of POD modes, $\{\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_n\}$. Expansion $\boldsymbol{q} \simeq \sum_{i=1}^{n} A_i(t)\boldsymbol{Q}_i$ and Galerkin projection of (1) onto the POD manifold yield a low n-dimensional ROM of the given system (an ODE in the time-dependent unknown amplitudes A_i).
It turns out that good POD manifolds to approximate both attractors and transients may result from snapshots coming from few numerical computations. In fact, it has been checked for various dissipative systems [5, 6, 7] that, (i) when dealing with transient dynamics, POD manifolds calculated from snapshots in a time interval $t_0 < t < t_1$ are also appropriate for larger values of t; (ii) when the equations depend on parameters, POD manifolds calculated for specific values of the parameters are also appropriate for other parameters values. In other words, the POD manifold is quite robust and weakly dependent on time and parameters, provided that some more POD modes than necessary are retained.
This is not surprising if we further assume that both $\boldsymbol{\mathcal{L}}^{-1}$ and $\boldsymbol{q} \to \boldsymbol{\mathcal{L}}^{-1}\boldsymbol{f}(\boldsymbol{q}, \mu)$ strongly splash the unit ball of $\boldsymbol{E}$. Indeed, the introduced assumptions *heuristically* imply that the (finite dimensional) attractors of (1) are well approximated by a low dimensional (say, of dimension n) manifold, $\boldsymbol{S} \subset \boldsymbol{E}$. Since $\boldsymbol{S}$ is low dimensional, a generic (for a specific value of μ) orbit of the system in a non-small time span is expected to cover all the dimensions of $\boldsymbol{S}$. Thus, if a set of $N > n$ snapshots is taken into account (being N sufficiently large), the associated POD manifold is expected to contain $\boldsymbol{S}$.
Good POD manifolds to efficiently compute the bifurcation diagram of (1), in a certain parameter span, can then be constructed from snapshots calculated by a time-dependent numerical solver in a generic time span and for a generic value of μ. Hence, the following method is proposed.

- Apply POD to a set of N snapshots associated with an orbit of (1) computed by a time-dependent numerical solver for a fixed value of μ, a generic initial condition, and a (non-small) time span.
- Select the first n POD modes, Galerkin-project (1) onto these modes, and construct the bifurcation diagram of the resulting ROM, which should approximate the bifurcation diagram of (1) if n is appropriately chosen.
- Results can be validated repeating calculations with a larger value of n and comparing the outputs.

Computational efficiency is further improved by using (in both POD and Galerkin projection) a non-standard inner product based on a limited number ($\sim 3n$) of mesh points among those in the spatial grid where snapshots are defined [1, 5, 7].

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RESULTS FOR THE GINZBURG-LANDAU EQUATION

The proposed method is applied to the *complex Ginzburg-Landau equation*

$$\partial_t u = (1 + \mathrm{i}\alpha)\partial^2_{xx} u + \mu u - (1 + \mathrm{i}\beta)|u|^2 u, \quad \text{with } \partial_x u = 0 \text{ at } x = 0, 1, \tag{2}$$

which is a well-known paradigm of a simple equation showing intrinsically complex dynamics [2]. This equation is invariant under the $D_1 \times SO(2)$ group $x \to 1-x$, $u \to u\,\mathrm{e}^{\mathrm{i}c}$. The latter symmetry and the Neumann boundary conditions allow for special solutions of type $u(x,t) = \mathrm{e}^{\mathrm{i}\delta t} u_0$, which can be (i) *monochromatic, spatially uniform periodic solutions* if $u_0 =$ constant; (ii) *monochromatic, spatially non-uniform periodic solutions* if $u_0 = u_0(x)$; and (iii) *quasi-periodic solutions* if $u_0 = u_0(x,t)$ is t-periodic, with a period incommensurable with $2\pi/\delta$ (which is generically expected). In addition, more complex quasi-periodic and chaotic solutions may appear if u_0 is quasi-periodic and chaotic, respectively. Some of them can be reflection symmetric (either instantaneously or in space-time), if u_0 is reflection symmetric.
The ROM is constructed as explained above, by selecting 8 POD modes calculated from 100 snapshots in $0 < t \leq 0.1$ obtained by integrating (2) with a Crank-Nicolson method for $\mu = 20$ and an initial condition which breaks reflection symmetry. The bifurcation diagram is computed (increasing μ) by continuing from the null solution at $\mu = 0$ and plotting the outward intersections of the orbits with a Poincaré hypersurface.
In order to illustrate the performance of the ROM, we select $\alpha = -10$ and $\beta = 1$, using μ as the bifurcation parameter. Note that $\alpha\beta < -1$, as required for the system to exhibit the *modulational instability* and thus complex behaviors. In fact, the associated bifurcation diagram exhibits more than ten bifurcations in the interval $0 < \mu \leq 300$ which involve solutions of types (i)–(iii) mentioned above, and a more complex quasi-periodic attractor (with three incommensurable frequencies) in the interval $132 < \mu < 144$. Some of them are reflection symmetry breaking bifurcations. The bifurcation diagram given by the ROM (omitted here) is plot indistinguishable from its counterpart obtained by integrating (2) with the (full) numerical solver, as are the associated attractors. The latter are illustrated in Figure 1 in four representative cases (from left to right): an attractor of type (i), an attractor of type (iii) exhibiting spatio-temporal reflection symmetry, an attractor of type (ii), and an attractor of type (iii) that shows instantaneous reflection symmetry.

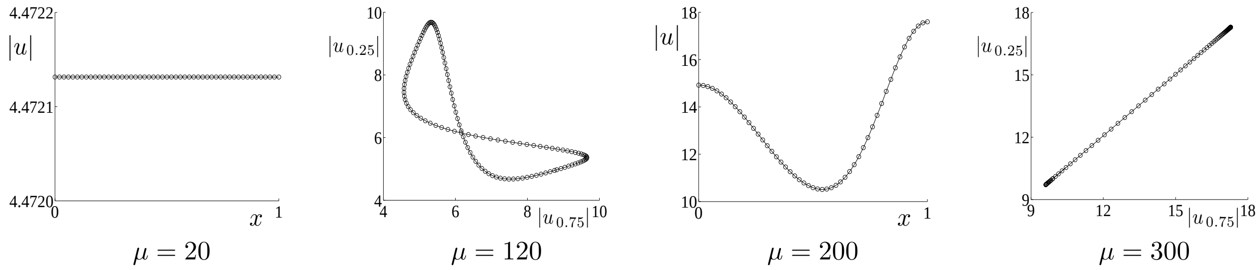

Figure 1. Solutions for the indicated values of μ: $|u|$ vs. x for solutions of types (i) and (ii), and $|u_{0.25}| \equiv |u(0.25,t)|$ vs. $|u_{0.75}| \equiv |u(0.75,t)|$ for solutions of type (iii), as provided by the numerical solver (plain circles) and the ROM (solid line).

The method is also tested in several additional cases. For instance

- Snapshots calculated for other values of μ (e.g. $\mu = 0$, retaining 8 modes) provide completely similar results.
- Instead of using a Crank-Nicolson scheme, snapshots can be calculated using a (rough) first-order semi-implicit numerical solver with a quite large time step. Retaining 8 modes, as above, provides completely similar results.
- Retaining 14 modes provides an approximation which is as good as that illustrated above, but in a much larger interval, namely in $0 < \mu \leq 2000$. In this case, the bifurcation diagram includes various chaotic windows.

Thus, it can be concluded that the method is able to calculate complex bifurcation diagrams in a quite robust, flexible, and fast way. An adaptive version of the method, which monitors the approximation errors and updates the POD manifold when needed, is the object of current research.

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