BUCKLING AND CUSP FORMATION IN GEL SWELLING AND TISSUE GROWTH

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Summary We popose of variational theory of gel swelling allowing the treatment of the buckling instability which occurs above some threshold defined by the amount of local volumetric growth. The plane strain patterns are well explained by the theory of bifurcations and the transition is supercritical in the idealized experimental cases. We explicit the singular role of surface tension and give an interpretation for the existence of cusps.

INTRODUCTION

The swelling of gels clamped to a substrate or to another soft material is still the subject of many theoretical, numerical and experimental investigations [1]. It also provides insights into the mechanical behavior of growing soft materials under geometrical constraints. Because swelling under constraints induces mechanical stresses, surface instabilities, and ultimately cusp formation, were observed experimentally. Starting from the neo-Hookean constitutive relationship together with the incompressibility requirement, typical of gels but also of biological tissues, we show that the standard equilibrium equations from continuum mechanics have a strong similarity with the two-dimensional Stokes flow [2]. We introduce a nonlinear stream function allowing an exact treatment of the incompressibility constraint. Our free energy formalism allows the extension both to arbitrary hyperelastic strain energies as well as additional interactions, such as surface energies. Assuming a homogeneous volumetric growth, we show that the threshold for a buckling instability to occur is completely governed by the overall volume increase. The determination of the wavelength at threshold, which scales as the initial thickness of the gel layer, requires to take surface effects into account. Our findings are extended to the weakly nonlinear case within the framework of bifurcation theory. Our theory is compared with experiments performed in a circular geomerty [3] .

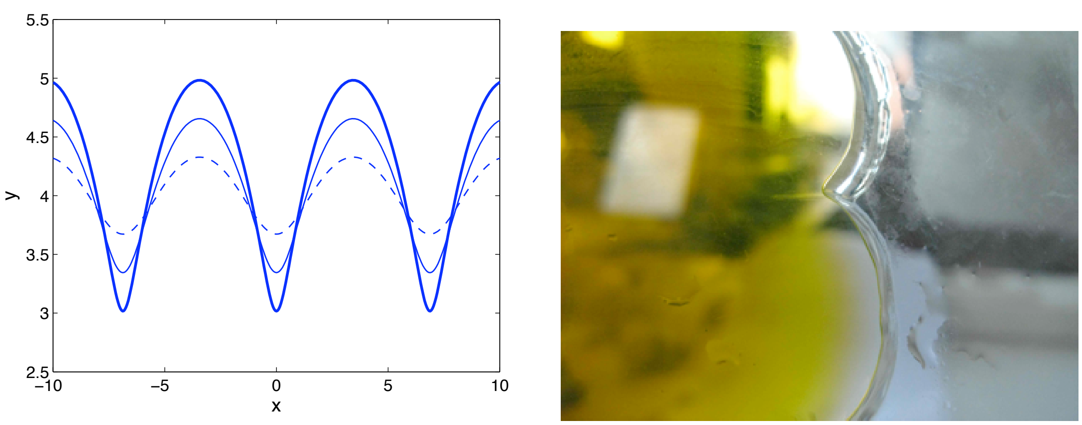

Figure 1. Surface profile obtained with the theory of perturbation applied to the stream-function. Note the asymmetry.Cusp exhibited in a awelling gel experiment in the circular geometry.

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Mechanical model and methods

Let us consider a soft body with thickness H (chosen as the unit of length) in the direction of the z-axis and lateral dimensions of order $L \gg H$. The volumetric growth model originally postulated for biological tissues [4] is adopted. We use the coordinate system $\mathbf{X}$=$\mathbf{X}(X, Y, Z)$ for the initial configuration, while $\mathbf{X}_g$=$\mathbf{X}_g(gX, Y, gZ)$ refers to the grown state and $\mathbf{x}$=$\mathbf{x}(x, y, z)$ to the current configuration [4]. The body is treated as an incompressible neo-Hookean material, with an elastic energy density $W = \frac{1}{2}\mu_s(\lambda_1^2 + \lambda_2^2 + \lambda_3^3 - 3)$, where μ_s is the shear modulus and the $\lambda_i(i = 1, 2, 3)$ are the principal stretches. The body is fixed at the bottom, while its top surface is free.

According to the volumetric growth theory, the deformation gradient $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ can be multiplicatively decomposed as: $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{G}$, where $\mathbf{F}_e$ denotes the elastic deformation tensor. $\mathbf{G}$ is the growth tensor, defined as $\mathbf{G} = g\mathbf{I}$, so that $J = det\mathbf{G} = g^2$. We get the primary buckling mode (i.e. stripes) along the x-axis, as studied by Ben Amar and Ciarletta [2]. Nevertheless the wavelength cannot be predicted preventing any nonlinear analysis. A non standard treatment is required to fix the wavelength via the introduction of a microscopic lengthscale which allows a nonlinear treatment.

Pattern transition from buckling to cusp

Although linear buckling assumes tiny deformations above threshold, the analysis via a stream-function gives a non symmetric and non harmonic surface deformation as shown in Fig.(1). It clearly suggests the possibility of cusp formation, which is confirmed via a weakly nonlinear analysis. This leads to an estimate of the volume variation necessary to induce cusp formation above the first buckling threshold. Despite our results, several theoretical points remain unanswered, such as the role of the microscopic length (arising from surface tension) on the nonlinearity as well as the nature of the cusp bifurcation. Can we consider the cusp appearance to be the nonlinear evolution of a regular stream function as suggested by a nonlinear analysis after the buckling transition or is it a subcritical bifurcation from the base state [5, 6].

Many experiments on gel swelling show surface instabilities forming periodic cusps. Folds and cusps arise above critical thresholds whose values are very close. It is why it remains unclear if buckling arises first and if cusps are simply nonlinear structures of a primary buckling instability. Only a theoretical treatment can give some insights to this theoretical question, important for the theory of bifurcations and patterning predictions [7].

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