TWISTED PLATEAU BORDERS AND THE COLLAPSE OF A MÖBIUS STRIP SOAP FILM

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Summary Among the classic problems in the study of minimal surfaces is the competition between solutions of different topology that span a given contour. Recent work on soap films spanning deformable wire frames has brought to the fore the dynamical processes that accompany the transition from one solution to another, specifically the topological transition from a one-sided surface (Möbius strip) to a two-sided one. We study this problem in detail, focusing on the 'twist singularity' associated with the change in the linking number between the film's Plateau border and the wire, and the twist-to-writhe conversion of the border as the frame is deformed. Geometrical aspects of the evolving surfaces are examined by means of a model of ruled surfaces spanning parametrized contours in comparison with minimal surfaces obtained numerically. These results explain how twist of the border generates experimentally observed light caustics and reveal the role of the finite size of the wire frame that supports the minimal surfaces.

INTRODUCTION

Ever since the work of Euler and Lagrange on functional extremization the problem of finding area-minimizing surfaces spanning prescribed contours has been recognised as a rich and challenging avenue for mathematical research. The most significant rigorous results are the solution of 'Plateau's problem', namely, the existence proofs by Rado and Douglas of minimal surfaces spanning a given contour. Because soap films spanning wire frames provide such a simple realisation of these surfaces much can be learned from experimentation with them. For instance, the possibility that there may be more than one minimal surface that can span a given contour was emphasized by Courant [1]. One of his most intriguing examples is the competition between the one-sided Möbius strip and a two-sided film. This can be visualized with a wire frame in the form of a double loop which, when removed from the soap solution, often possesses a second disk-shaped surface that can be destroyed to leave the Möbius strip (Figure 1). Smoothly untwisting the loops renders that minimal surface unstable, with a rapid transition to the two-sided solution.

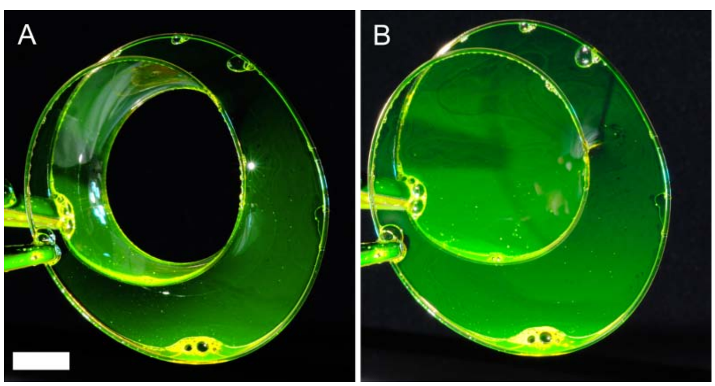

Figure 1. Topological transition of a soap film Möbius strip [2]. As its frame is gradually distorted, a critical point is reached at which the one-sided film (A) transforms rapidly to a two-sided solution (B). Scale bar in (A) is 2 cm.

A SUBTLE TOPOLOGICAL TRANSITION

The instability of the Möbius strip soap film as its frame is deformed is reminiscent of the loss of stability of the catenary that spans two parallel circular wires as they are pulled apart. Experiments show that the collapse of a catenary preserves the axisymmetry of the initial condition and produces a pinching singularity on axis, away from the boundary. Extensive theoretical studies of inviscid collapse dynamics have shown excellent agreement with these observations. Related work on submerged air bubble pinchoff has shown the possibility of broken axisymmetry in a system which is axisymmetric in the far field, but with singularities still occurring away from boundaries.
At first glance, the transition in Figure 1 appears to be merely a non-axisymmetric version of the catenary collapse, the two looped sections of the wire frame playing the role of the separate rings holding the catenary. Such is not the case.

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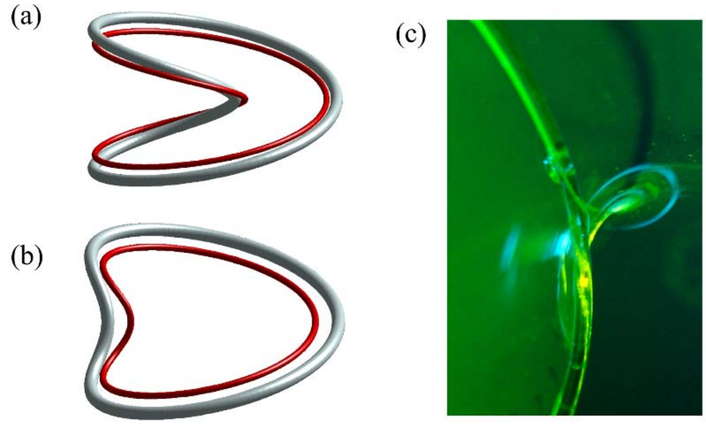

Figure 2. Schematic illustration of linking of Plateau border (red) and wire frame (gray) for (a) the Möbius strip topology and (b) the two-sided solution. (c) Caustic visible from highly-curved surface after the collapse of the Möbius strip surface [2].

In a recent paper [2] we reported the discovery that there is a 'twist singularity' associated with the transition that occurs at the wire frame bounding the soap film. This can be understood as a consequence of the fact illustrated in Figures 2a and b that the Plateau border and the wire frame are doubly-linked in the Möbius confiuration and unlinked for the two-sided surface. Thus, the film collapse involves *reconnection* of the Plateau border that snakes along the frame. After the transition there is a caustic visible due to the high curvature of the film in the neighborhood of the collapse point.

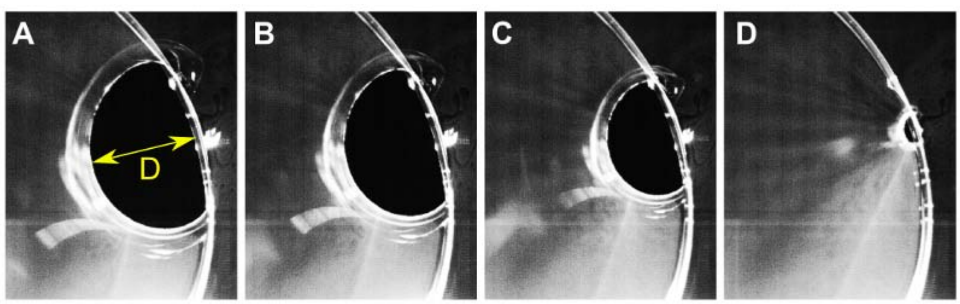

Figure 3. Collapse of the Möbius strip 'throat' [2]. (A-D) Frames from a high-speed movie at 5.4 ms intervals show the approach to the singularity and collision with the frame. The throat diameter D used to quantify the collapse dynamics is indicated in A.

Since the linking number of the Plateau border and the wire frame is a quantity defined at the wire frame, it is perhaps not surprising that the singularity associated with reconnection is driven to occur at the frame. This is shown in Fig. 3 in frames from a high-speed movie. The throat diameter D exhibits an intriguing crossover in power-law behavior as it collapses to the wire, asymptotically displaying the $2/3$ exponent associate with a balance of surface tension and inertia. The early-time behavior appears consistent with a balance involving viscous dissipation in the coating film on the wire.

Together these observations help define a new class of mathematical and physical problems in minimal surfaces. Building on our earlier work [2] we shall report in this talk on more recent experimental and theoretical studies of the twisting of Plateau borders in the context of topological rearrangements of minimal surfaces.

References

[1] Courant R (1940) Soap film experiments with minimal surfaces. *Amer Math Mon* 47:167-174.

[2] Goldstein, R.E., Moffatt, H.K., Pesci, A.I., and Ricca, R.L. Soap-film Möbius strip changes topology with a twist singularity, *Proc. Natl. Acad. Sci. USA* **107**, 21979 (2010).