# ATTRIBUTES OF HYPERBOLIC ATTRACTORS IN NON-SMOOTH MECHANICAL OSCILLATORS

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*Summary* During the last few decades there appeared several proposals of dynamical systems which possess hyperbolic or quasi-hyperbolic properties. Classical known examples of hyperbolic attractors are artificial mathematical constructions. However, recently there appeared also examples of real physical systems (electronic circuits) exhibiting hyperbolic properties. The topic of this paper is a comparative summary of typical characteristics of hyperbolic attractor (or quasi-hyperbolic) with the properties of attractor of non-smooth mechanical systems, i.e., frictional or impact oscillators. The presented analysis of driven and self-excited friction oscillator shows that the main common feature of these attractors is their structural stability (in spite of their chaotic nature), which is manifested by a low sensitivity of system dynamics and structure of the phase-space for changes of parameters in differential equations.

## CONCEPT OF HYPERBOLICITY

One of the more interesting aspects of mathematical chaos theory is the concept of hyperbolicity [1-4,7,9]. According to the existing definitions [1], the hyperbolic chaotic attractor, and strictly speaking uniformly hyperbolic, consists of a saddle orbit solely, where stable and unstable directions are clearly defined. In addition, stable and unstable manifolds are the same size and they are not tangent to each other in phase space, i.e., they intersect only transversally. Therefore the hyperbolic attractor is structurally stable, in spite of its chaotic nature. This property manifests with a low sensitivity of system dynamics and the structure of the phase-space on changes (at least modest) of the system parameters. It is also possible the existence of the so-called quasi-hyperbolic attractors, in which mathematical criterions of the hyperbolicity are not so strictly fulfilled [1].

The best known examples of uniformly hyperbolic attractors are artificial constructions of the mathematical form representations of differential equations, such as Smale-Williams solenoid or Plykhin attractor [1-3, 9]. It seems that the mathematical theory of hyperbolic chaos had not been successfully applied to real physical object, although the concepts of this theory are widely used to interpret the actual behavior of chaotic nonlinear systems. Significant progress in the search of hyperbolicity in physical systems has been made only in recent years by Kuznetsov. He designed a simple example not autonomous phase stream, which clearly manifests a strange, hyperbolic attractor [6].

The existence of hyperbolic attractors of mechanical systems has been partially confirmed (this is only a conjecture) in one case [5], but this solution is not useful in the context of applications. However, the detailed numerical analysis of non-smooth mechanical systems, especially classical stick-slip friction oscillator with harmonic driving (see Figure 1) revealed the structural stability of the system response in ranges of the chaotic motion, as in the case of a typical hyperbolic solution.

## NUMERICAL EXAMPLE

One of the mechanical systems, which is suspected to have hyperbolic-like properties is a typical forced stick-slip mechanical oscillator [8] shown in Figure 1, composed of mass located on the conveyor belt moving with the constant speed $v_B$. The dynamics of such system is described by the following second order differential equation:

$$m\ddot{x} = -c[x - U_0 \cos(\Omega t)] + F_N f(v_r)\,, \tag{1}$$

where $m$ – mass analyzed oscillator [kg], $c$ – spring stiffness [N/m], $\Omega$ – forcing frequency [s$^{-1}$], $U_0$ – forcing amplitude [m], $F_N$ – normal pressure force [N], $v_r = (v_B - \dot{x})$ – relative velocity [m/s], $f(v_r)$ – function describing friction characteristic. During the numerical analysis, classical Coulomb model of friction has been taken under consideration, i.e.,

$$f(v_r) = \begin{cases} f_S & v_r = 0, \\ f_C \operatorname{sgn}(v_r) & v_r \neq 0. \end{cases} \tag{2}$$

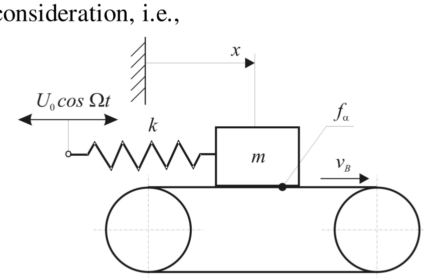


**Figure 1.** Self-exited oscillator with harmonic forcing.

The results of numerical calculations are presented in Figure 2a and 2b. In Figure 2a we can see a bifurcation diagram in 3D perspective that allows to observe the insensitivity of the system attractor structure on the change of the control parameter in the chaotic range. By contrast, in Figure 2b the course of the largest Lyapunov exponent in the same range of control parameter $F_N/c$, is shown.
Calculation of Lyapunov exponents of the discontinuous systems with friction requires the use of numerical algorithms that allow passage through the frictional stick-slip discontinuity. In the present case, this exponent has been estimated using modificated synchronization method of the Lyapunov exponent estimation, which was previously developed by the authors [10].

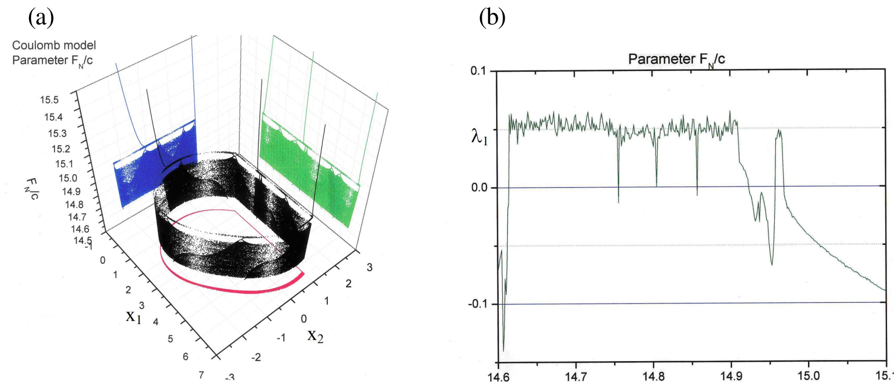


**Figure 2**. Bifurcation diagram of the system (1) with Coulomb friction in 3D view (a) and the course of the corresponding largest Lyapunov exponent (b) in the same range of control parameter ($F_N/c$).

## CONCLUSIONS

On the basis of the property of hyperbolic attractors, and numerical analysis of harmonically forced stick-slip oscillator we can conclude that this system is not hyperbolic in the sense of the classical theory of homogeneous hyperbolicity. Important feature of the hyperbolic attractors is their structural stability when control parameter changes. It causes that the value of positive Lyapunov exponent is also not very sensitive to these changes. The largest Lyapunov exponent does not change significantly (it fluctuates around the value of 0.05 in Figure 2b) in considered interval of the chaotic motion. This fact confirms the stability of the torus structure observed in Figure 2a. These results were a base for our conjecture on at least quasi-hyperbolic nature of the analyzed friction oscillator. Unfortunately, detailed numerical verification excluded any hyperbolic features in considered case. On the other hand, we have shown that the structural stability observed in analyzed case is a specific property of discontinuous stick-slip dynamics of the system.

Summing up, these results provide some grounds to consider that the attractor of stick-slip friction oscillator has characteristics similar to those of quasi-hyperbolic attractor.

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**References**

[1] Afraimovich V. S., Bykov V. V., Shilnikov L. P.: Dokłady Akademii Nauk 234, 336 (1977), in Russian.
[2] Anishchenko V. S., Astakhov V. V., Neiman A. B., Vadisova T. E., Schimansky-Geier L.: Nonlinear dynamics of Chaotic and Stochastic Systems, Springer, Berlin, Heidelberg (2002).
[3] Belykh V., Belykh I., Mosekilde E.: Hyperbolic Plykin attractor can exist neuron models, International Journal of Bifucation and Chaos 15, 3567 (2005).
[4] Eckmann J.-P., Ruelle D.: Ergodic theory of chaos and strange attractors, Review of Modern Physics 57, 617 (1985).
[5] Hunt T. J., MacKay R. S.: Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor, Nonlinearity 16, 1499 (2003).
[6] Kuznetsov S. P.: Example of a Physical System with a Hyperbolic Attractor of the Smale-Williams Type, Physical Review Letters 95, 144101 (2005).
[7] Ott E.: Chaos in Dynamical Systems, Cambridge University Press, Cambridge (1993).
[8] Popp K. and Stelter P.: Non-linear oscillations of structures induced by dry friction, in Non-linear Dynamics in Engineering Systems, (1990), ed. W. Schiehlen, Springer, New York.
[9] Smale S.: Differentiable dynamical systems, Bulletin of the American Mathematical Society, 73, 1967, s. 747-817.
[10] Stefański A., Kapitaniak T.: Estimation of the dominant Lyapunov exponent of non-smooth systems on the basis of maps synchronization. *Chaos, Solitons & Fractals*, 15, 2003, 233-244.