DISENTANGLING MULTI-LEVEL SYSTEMS: AVERAGING, CORRELATIONS, MEMORY

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Summary We consider two weakly coupled systems and adopt a perturbative approach based on the Ruelle response theory to study their interaction[1]. We propose a systematic way to parameterize the effect of the coupling as a function of only the variables of the system of interest. Our focus is on describing the impacts of the coupling on the long-term statistics. At first order, the coupling can be surrogated by adding a deterministic perturbation to the autonomous dynamics of the system of interest. At second order, there are two separate contributions. One term takes into account the fluctuations in the coupling, which can be parameterized as a stochastic forcing with given spectral properties. The other one is a memory term, coupling the system of interest to its previous history. This effect can be parameterized as a perturbation with memory on the single system. We emphasize that our results do not rely on assuming a time scale separation, and, if such a separation exist, can be used equally well to study the statistics of the slow as well as that of the fast variables. By recursively applying the technique proposed here, we can treat the general case of multilevel systems.

OVERVIEW

We consider a weakly coupled multi-level dynamical system:

$$\dot{X} = F_X(X) + \Psi_X(X,Y), \tag{1}$$

$$\dot{Y} = F_Y(Y) + \Psi_Y(X,Y). \tag{2}$$

We want to derive a reduced dynamics for the X variables, where the coupling with the Y variables is parametrized:

$$\dot{X} = F_X(X) + \tilde{\Psi}_X(X,t) \tag{3}$$

Numerical simulations of multi-level dynamical systems (see e.g.[2]) have shown that the inclusion of a noise term in the parametrization improves the quality of long-term statistics. The following questions arise:

- Why does correlated noise improve long-term statistics?
- Can the parameters (intensity, correlations) be analytically derived?
- Is this a complete description or can additional terms improve the statistics?

This problem was discussed by Hasselmann [3] and put into a rigorous setting of the averaging method by Arnold. This method makes use of a separation of time scales among the levels of the system. Here we don't make use of such a separation and make use of a weak coupling limit instead.

RUELLE RESPONSE THEORY

In order to analyze this, we make use of a response formula derived by Ruelle [4]. This formula describes the variation of long-term averaged observables as the underlying dynamical system is altered. The long-term averages can be written as: $\rho_0(A) = \lim_{T\to\infty} \frac{1}{T}\int_0^T d\tau \int l(dx) A(f^\tau(x))$. They depend on the dynamical system through the flow f^τ, which describes the position of a point x after a time τ. Ruelle's formula gives an expansion of a perturbed measure $\tilde{\rho}$ in terms of the unperturbed measure ρ. This can be considered as a non-equilibrium extension of the fluctuation-dissipation theorem (FDT) [5]. The layout of our method is as follows:

- Treat the coupling Ψ as a perturbation and write the coupled measure $\tilde{\rho}$ in terms of the uncoupled measure ρ.
- Use the fact that the uncoupled measure is a product measure on X and Y to write the response to the coupling as a response of the X system, where the response functions are determined by the structure of correlations in Y.
- Construct a parametrization that results in the same response when added to the dynamics

If the dynamics of a phase-space point is given by $\dot{x} = F(x)$, then the time derivative of an observable A is given by applying the Liouville operator $\mathcal{L}_0 = F\cdot\nabla$. A perturbation of F by Ψ corresponds to changing the operator $\mathcal{L}_0$ to $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 = F\cdot\nabla + \Psi\cdot\nabla$. By making use of the Dyson expansion, the time evolution of an operator, which is given by $\exp(\mathcal{L}t)$ can be expanded around $\Pi_0(t) := \exp(\mathcal{L}_0 t)$ in orders of $\mathcal{L}_\infty$. We obtain:

$$\begin{aligned}\tilde{\rho}(A) &= \rho_0(A) + \int_0^\infty ds_1 \rho_0(\mathcal{L}_1\Pi_0(s_1)A) + \int_0^\infty ds_1 \int_0^{s_1} ds_2 \rho_0(\mathcal{L}_1\Pi_0(s_1)\mathcal{L}_1\Pi_0(s_2)A) + O(\Psi^3)\\ &= \rho_0(A) + \delta\rho^{(1)}(A) + \delta\rho^{(2)}(A) + O(\Psi^3)\,,\end{aligned} \tag{4}$$

The first order term is given by an integral of the impact of an infinitesimal perturbations over all past times. The second order term is calculated by integrating all impacts of infinitesimal perturbations at two different times in the past.

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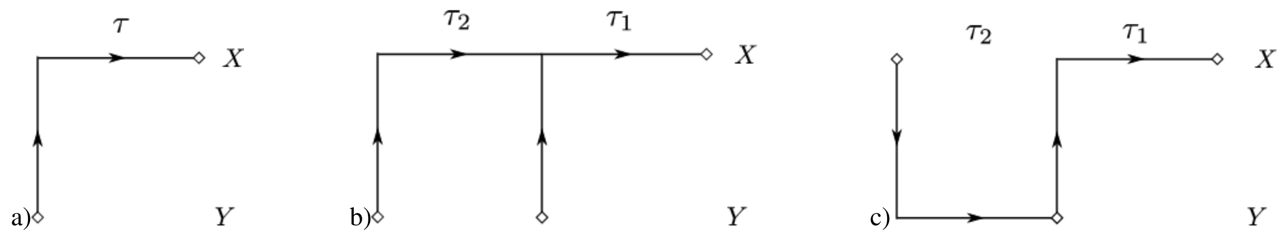
The unperturbed Liouville operator is explictly written as $\mathcal{L}_0 = F_X(X) \cdot \nabla_X + F_Y(Y) \cdot \nabla_Y$ while its perturbation is expressed as $\mathcal{L}_1 = \Psi_X(Y) \cdot \nabla_X + \Psi_Y(X) \cdot \nabla_Y$. Even if we have treated the general case, for simplicity we stick here to couplings where $\Psi_X(X,Y) = \Psi_X(Y)$ and $\Psi_Y(X,Y) = \Psi_Y(X)$. We calculate that the response to the coupling is given by three terms. The response functions can be diagrammatically represented as in the figure below.



As prescribed by Ruelle's response formula, the different orders in the response are given by combinations of interactions at different times in the past. In formulas, these terms give the following first and second order Green's functions:

$$\text{a) } G_A^{(1)}(\tau) = \int \rho_{0,X}(dX) M \cdot \nabla_X A(f_X^\tau(X)), \quad M = \langle \Psi_X(Y) \rangle_{\rho_{0,Y}} \tag{5}$$

$$\text{b) } G_{A,f}^{(2)}(\tau_1,\tau_2) = \int \rho_{0,X}(dX) g(\tau_1) \partial_X \partial_{f_X^{\tau_1}(X)} (A \circ f_X^{\tau_2})(f_X^{\tau_1}(X)),\ g(\tau_1) = \langle \Psi'_X(Y) \Psi'_X(f_Y^{\tau_1}(Y)) \rangle_{\rho_{0,Y}},\ \Psi'_X(Y) = \Psi_X(Y) - M \tag{6}$$

$$\text{c) } G_{A,m}^{(2)}(\tau_1,\tau_2) = \int \rho_{0,X}(dX) h_j(\tau_2, X) \partial_{f^{\tau_2}(X),j} (A(f^{\tau_1+\tau_2}(X)),\ h_j(\tau, X) = \langle \Psi_{Y,i}(X) \partial_{Y,i} \Psi_{X,j}(f^\tau(Y)) \rangle_{\rho_{0,Y}} \tag{7}$$

These terms can be interpreted as follows:

- M corresponds to an averaged version of the instantaneous coupling Ψ_X;
- g describes the second order fluctuations of the coupling Ψ_X around the average value M;
- h is an integral kernel describing the delayed effect of X on itself through the coupling with Y.

PARAMETRIZATION

We show that the different terms in the response can be parametrized by three processes

- A constant forcing M that corresponds to the response function $G_A^{(1)}$;
- A stochastic forcing $\sigma(t)$ that mimics the correlations contained in $G_{A,f}^{(2)}$: $\langle \sigma(t_2)\sigma(t_1) \rangle = g(t_2 - t_1)$ and $\langle \sigma(t) \rangle = 0$;
- A memory forcing with kernel h taking into account the delayed effect that is present in $G_{A,m}^{(2)}$

As parameterization we finally get (in the general case M is X dependent and the noise term is multiplicative):

$$\dot{X}(s) = F_X(X(s)) + M + \sigma(s) + \int_0^\infty d\tau\, h(s-\tau, X(t-\tau))\,. \tag{8}$$

CONCLUSIONS

- We find an explanation for the improvement of long-term statistics upon adding correlated noise.
- We show how the parameters of the noise process can be calculated from the correlations of the unresolved process.
- We show that a memory term should also be added to the parameterization, specifically when the time-scales in the resolved and unresolved processes are not well separated (i.e. when the averaging method does not apply).
- Our method bears resemblance with the Mori-Zwanzig projection operator formalism [6]

References

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