GEOMETRIC INTERPRETATION OF DISCRETIZED SYMMETRIC-CONSERVATIVE METRIC FOR HIGHER-ORDER FINITE-DIFFERENCE SCHEME

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Summary The new evaluations of time metrics and Jacobian are introduced for any higher-order finite-difference method employed for the computation of fluid motion around the arbitrary body shape. They are based on two analytical forms: *asymmetric-* and *symmetric-conservative forms* whose discretizations preserve the freestream on every time step for any higher-order finite difference method. The computational results show the validation of both of two evaluations, and the *symmetric-conservative forms* give better results in terms of resolution and robustness. The reason that the *symmetric-conservative forms* for time metrics and Jacobian give better results is pursued in view of their geometric meanings. It is shown that the discretized *symmetric-conservative form* of Jacobian corresponds to the weighted average of corresponding volumes around the specific point among the cell considered; the similar interpretation can be applied for *symmetric-conservative form* of time metrics.

INTRODUCTION

The body-fitted coordinate system is often adopted when fluid motion around the arbitrary body shape is computed. In this case, the coordinate-transform metrics from body-fitted coordinate system into Cartesian coordinate system are introduced for the computation. Although transform metrics analytically satisfy the freestream preservation which is so-called “geometric conservation law" (GCL)[2], some discretized forms of metrics break the GCL identities with the use of finite-difference methods. The GCL identities consist of “surface closure law" (SCL) and “volume conservation law" (VCL)[6], and the VCL identity is focused in this paper. Some techniques for the discretization of time metrics and Jacobian were proposed in terms of geometries, e.g., finite-volume methods, which is not suitable for higher-order finite-difference methods. In this paper, the analytical form which those classical discretizations are based on is called *non-conservative form* for time metrics and Jacobian. We propose new analytical forms of time metrics and Jacobian: *asymmetric-* and *symmetric-conservative forms*, whose discretizations satisfy the VCL identity on every time step for any higher-order finite difference method. The computation of Euler equations is carried out for comparison of *asymmetric-* and *symmetric-conservative forms* for time metrics and Jacobian; then, the difference between these new forms is pursued in terms of resolution and robustness. Finally, the geometric interpretation of *symmetric-conservative forms* for time metrics and Jacobian is introduced for any higher-order finite-difference method, and it is shown to be the extension of the classical discretizations based on finite-volume method.

PROBLEMS

New evaluation of time metrics and Jacobian

The Cartesian coordinate (t, x, y, z) and body-fitted coordinate (τ, ξ, η, ζ) are introduced under the assumption $t = \tau$. In the following, the expression of partial differentiation is simplified by subscripts: for example, time metric $\partial\zeta/\partial t$ is expressed as ζ_t. Analytical expressions of ζ_t/J and J are

$$(\textit{Non-cons.})\quad \zeta_t/J = x_\eta y_\tau z_\zeta - x_\eta y_\zeta z_\tau + x_\zeta y_\eta z_\tau - x_\zeta y_\tau z_\eta + x_\tau y_\zeta z_\eta - x_\tau y_\eta z_\zeta, \tag{1}$$

$$(\textit{Non-cons.})\quad 1/J = x_\xi y_\eta z_\zeta - x_\eta y_\xi z_\zeta + x_\zeta y_\xi z_\eta - x_\xi y_\zeta z_\eta + x_\eta y_\zeta z_\xi - x_\zeta y_\eta z_\xi, \tag{2}$$

or

$$(\textit{Asym-cons.})\quad \zeta_t/J = [\{(x_\eta y)_\xi - (x_\xi y)_\eta\}z]_\tau + [\{(x_\tau y)_\eta - (x_\eta y)_\tau\}z]_\xi + [\{(x_\xi y)_\tau - (x_\tau y)_\xi\}z]_\eta, \tag{3}$$

$$(\textit{Asym-cons.})\quad 1/J = [\{(x_\xi y)_\eta - (x_\eta y)_\xi\}z]_\zeta + [\{(x_\zeta y)_\xi - (x_\xi y)_\zeta\}z]_\eta + [\{(x_\eta y)_\zeta - (x_\zeta y)_\eta\}z]_\xi, \tag{4}$$

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where J is Jacobian determinant of coodinate transformation; *Non-cons.* and *Asym-cons.* denote *non-* and *asymmetric-conservative form*, respectively. The *symmetric-conservative forms* can be introduced in terms of coordinate invariance:

$$
\begin{aligned}
(\textit{Sym-cons.})\quad 1/J = \frac{1}{3}\Bigg\{ & \left[\frac{1}{2}\left\{(x_\xi y)_\eta - (x_\eta y)_\xi + (y_\eta x)_\xi - (y_\xi x)_\eta\right\} z\right]_\zeta \\
& + \left[\frac{1}{2}\left\{(y_\xi z)_\eta - (y_\eta z)_\xi + (z_\eta y)_\xi - (z_\xi y)_\eta\right\} x\right]_\zeta + \left[\frac{1}{2}\left\{(z_\xi x)_\eta - (z_\eta x)_\xi + (x_\eta z)_\xi - (x_\xi z)_\eta\right\} y\right]_\zeta \\
& + \left[\frac{1}{2}\left\{(x_\zeta y)_\xi - (x_\xi y)_\zeta + (y_\xi x)_\zeta - (y_\zeta x)_\xi\right\} z\right]_\eta + \left[\frac{1}{2}\left\{(y_\zeta z)_\xi - (y_\xi z)_\zeta + (z_\xi y)_\zeta - (z_\zeta y)_\xi\right\} x\right]_\eta \\
& + \left[\frac{1}{2}\left\{(z_\zeta x)_\xi - (z_\xi x)_\zeta + (x_\xi z)_\zeta - (x_\zeta z)_\xi\right\} y\right]_\eta + \left[\frac{1}{2}\left\{(x_\eta y)_\zeta - (x_\zeta y)_\eta + (y_\zeta x)_\eta - (y_\eta x)_\zeta\right\} z\right]_\xi \\
& + \left[\frac{1}{2}\left\{(y_\eta z)_\zeta - (y_\zeta z)_\eta + (z_\zeta y)_\eta - (z_\eta y)_\zeta\right\} x\right]_\xi + \left[\frac{1}{2}\left\{(z_\eta x)_\zeta - (z_\zeta x)_\eta + (x_\zeta z)_\eta - (x_\eta z)_\zeta\right\} y\right]_\xi \Bigg\}, \quad (5)
\end{aligned}
$$

where *Sym-cons.* denote *symmetric-conservative form*; similarly, time metrics can be expressed in *symmetric-conservative forms* which are omitted for brevity. The computation of Euler equations are carried out on three-dimensional moving and deforming meshes with 6th-order compact-differencing scheme to validate these new forms for time metrics and Jacobian; first, the uniform flow is calculated for the preservation error of conservative quantities; second, two-dimensional isentropic vortex is calculated. Those results indicate the effectiveness of both of new evaluations, i.e., *asymmetric-* and *symmetric-conservative forms* for time metrics and Jacobian. It is also shown that the resolution and robustness with the use of *symmetric-conservative forms* are better than those with the use of *asymmetric-conservative forms*.

Geometric interpretations of symmetric-conservative forms

The *symmetric-conservative form* of Jacobian analytically corresponds to the following expression:

$$
\begin{aligned}
1/J &= \frac{1}{3}\Big\{(\boldsymbol{S}^\zeta\cdot\boldsymbol{r})_\zeta + (\boldsymbol{S}^\eta\cdot\boldsymbol{r})_\eta + (\boldsymbol{S}^\xi\cdot\boldsymbol{r})_\xi\Big\} \quad (6)\\
&= \frac{1}{3}\Big\{\underline{(\boldsymbol{S}^\zeta)_\zeta\cdot\boldsymbol{r} + (\boldsymbol{S}^\eta)_\eta\cdot\boldsymbol{r} + (\boldsymbol{S}^\xi)_\xi\cdot\boldsymbol{r}} + \boldsymbol{S}^\zeta\cdot\boldsymbol{r}_\zeta + \boldsymbol{S}^\eta\cdot\boldsymbol{r}_\eta + \boldsymbol{S}^\xi\cdot\boldsymbol{r}_\xi\Big\} \quad (7)
\end{aligned}
$$

where $\boldsymbol{S}^\zeta$ indicates the area vector of $\zeta = const$ plane; $\boldsymbol{r}$ denotes the position vector in physical space. The three terms underlined in Eq.(7) analytically vanish due to the SCL identities, and the other three terms correspond to *non-conservative form* for Jacobian. Note that the *non-conservative form* directly gives the geometric interpolation of Jacobian in finite-volume fashion. The discretized *symmetric-conservative form* for Jacobian can be deformed similarly to Eq. (7), and it is shown that its discretization is identical with the weighted average of corresponding volumes around the specific point among the cell considered.

CONCLUSION

The new evaluations for time metrics and Jacobian are introduced when any higher-order finite-difference method is employed for the computation of fluid motion. They are based on two analytical forms: *asymmetric-* and *symmetric-conservative forms*, respectively. The computational results show the validation of both of two evaluations, and the *symmetric-conservative forms* give better results in terms of resolution and robustness. The reason why the *symmetric-conservative forms* for time metrics and Jacobian give better results is pursued in view of their geometric meanings. It is shown that when any higher-order finite-difference method is employed for the discretization, the discretized *symmetric-conservative form* of Jacobian corresponds to the weighted average of corresponding volumes around the specific point among the cell considered; the similar interpretation can be applied for *symmetric-conservative form* of time metrics.

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