SECOND ORDER ACCURATE DERIVATIVES AND INTEGRATION SCHEMES FOR MESHFREE METHODS

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Summary The consistency condition for the nodal derivatives in traditional meshfree Galerkin methods is only the differentiation of the approximation consistency (DAC). One missing part is the consistency between a nodal shape function and its derivatives in terms of the divergence theorem in numerical forms. A consistency framework including the DAC and the discrete divergence consistency (DDC) is proposed, based on which a quadratically consistent 3-point (QC3) integration scheme is developed for quadratic approximations.

INTRODUCTION

In meshfree Galerkin methods, such as the element-free Galerkin (EFG) method [1], the derivatives of the nodal shape functions are computed by the differentiation of the approximation consistency (DAC). However, the meshfree nodal shape function, as a smooth function, has a consistency requirement with its derivatives in terms of the divergence theorem. Krongauz and Belytschko [2], in meshfree context, first pointed out such requirement and proposed consistent pseudo-derivatives by using Petrov-Galerkin discretizations. However, their formulation only considered linear approximations and does not meet such requirement exactly in a numerical sense. Bonet and Kulasegaram [3] proposed an integration correction for linear approximations with an iterative procedure to numerically enforce such requirement. Chen *et al.* [4] formulated this requirement in numerical form as an integration constraint for linear approximations and proposed an elegant strain smoothing technique which meets the integration constraint exactly. They named their method stabilized conforming nodal integration (SCNI) and showed that SCNI is stable and more efficient than meshfree methods with standard Gauss quadratures.
The purpose of this paper is to propose a framework for the consistency of the meshfree nodal derivatives including the DAC and the divergence theorem requirement; the latter is formulated and named as a discrete divergence consistency (DDC). Both the DAC and the DDC are employed as the necessary conditions to be met by building corrected (smoothed) derivatives of the nodal meshfree shape functions. In addition, a novel quadratically consistent 3-point (QC3) integration scheme with corrected nodal derivatives using background triangle elements is developed in the proposed consistency framework and extends the previous methods [2-4] from linear to quadratic approximations.

THE PROPOSED FRAMEWORK FOR THE CONSISTENCY OF NODAL DERIVATIVES

As mentioned above, the proposed consistency framework includes the DAC and the DDC. The DAC is as usual and can be written as

$$\mathbf{p}_{,i}(\mathbf{x}) = \sum_I \mathbf{p}(\mathbf{X}_I) N_{I,i}(\mathbf{x}) \tag{1}$$

where $\mathbf{p}(\mathbf{x})$ is the vector of base functions used in the construction of meshfree approximations. On the other hand, the meshfree nodal shape functions $N_I(\mathbf{x})$, as a smooth function, has a consistency with its derivatives $N_{I,i}(\mathbf{x})$ in terms of the divergence theorem and such condition can be written as

$$\int_{\Omega_S} N_{I,i}(\mathbf{x}) f(\mathbf{x}) \mathrm{d}\Omega = \int_{\Gamma_S} N_I(\mathbf{x}) f(\mathbf{x}) n_i \mathrm{d}\Gamma - \int_{\Omega_S} N_I(\mathbf{x}) f_{,i}(\mathbf{x}) \mathrm{d}\Omega \tag{2}$$

where $f(\mathbf{x})$ is an arbitrary smooth function and Ω_S bounded by Γ_S is a sub-domain where $N_I(\mathbf{x})$ has a influence. Theoretically, Eq.(2) should be met for any smooth function $f(\mathbf{x})$. However, in practice, such rigorous condition can not be reached and it is necessary only for some smooth function closely related to the problem under study. For the elastostatic problem, the set of key functions $f(\mathbf{x})$ can be determined by an analysis of the integration by parts in the derivation of the weak form from the equilibrium equation (strong form) as follows

$$\int_{\Omega_S} N_{I,j}(\mathbf{x}) \sigma_{ij}(\mathbf{x}) \mathrm{d}\Omega = \int_{\Gamma_S} N_I(\mathbf{x}) \sigma_{ij}(\mathbf{x}) n_j \mathrm{d}\Gamma - \int_{\Omega_S} N_I(\mathbf{x}) \sigma_{ij,j}(\mathbf{x}) \mathrm{d}\Omega \tag{3}$$

If the displacement fields are approximated with the base $\mathbf{p}(\mathbf{x})$, the stress fields $\sigma_{ij}(\mathbf{x})$ fall into the space spanned by the base functions $\mathbf{q}(\mathbf{x}) = \mathbf{p}_{,x}(\mathbf{x}) \cup \mathbf{p}_{,y}(\mathbf{x})$. Therefore, the set of key functions $f(\mathbf{x})$ are $\mathbf{q}(\mathbf{x})$ if we compare Eq.(2) with Eq.(3), and Eq.(2), after the integration is performed numerically, can be written as

$$\sum_{H=1}^{m_H} W_H N_{I,i}(\mathbf{x}_H) \mathbf{q}(\mathbf{x}_H) = \sum_{L=1}^{k_L} \sum_{G=1}^{m_G} N_I(\mathbf{x}_G) n_i^L w_G - \sum_{H=1}^{m_H} W_H N_I(\mathbf{x}_H) \mathbf{q}_{,i}(\mathbf{x}_H) \tag{4}$$

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Eq.(4) presents a consistency requirement for computing the derivatives and it is named discrete divergence consistency (DDC) which together with the DAC, i.e. Eq.(1), constitutes the proposed consistency framework.

QUADRATICALLY CONSISTENT 3-POINT (QC3) INTEGRATION SCHEME

The scheme uses background triangle elements with 3 evaluation points per element to perform the domain integration. For quadratic approximations $\mathbf{p}(\mathbf{x})=\begin{bmatrix}1 & x & y & x^2 & xy & y^2\end{bmatrix}^{\mathrm{T}}$, $\mathbf{q}(\mathbf{x})=\begin{bmatrix}1 & x & y\end{bmatrix}^{\mathrm{T}}$ and therefore Eq.(4) essentially provides three equations which can be used to determine the nodal derivatives at these 3 evaluation points in one element. One can easily prove that such "corrected" derivatives directly computed by Eq.(4) also meet Eq.(1) exactly for quadratic approximations provided that the boundary integration in Eq.(4) is performed by 2 Gauss points per edge of the background triangle elements. The proof is omitted here due to the page limitation. It is noted that, for linear approximation $\mathbf{p}(\mathbf{x})=\begin{bmatrix}1 & x & y\end{bmatrix}^{\mathrm{T}}$, $\mathbf{q}(\mathbf{x})=[1]$ and Eq.(4) can be employed to compute the derivatives directly if only one evaluation point per element is used. This corresponds to the SCNI [4] and it is named the linear consistent 1-point (LC1) integration method. Note that LC1, or the SCNI, is only linearly consistent even if the quadratic base is used for the approximations since only the linear DDC, i.e. Eq.(4) with $\mathbf{q}(\mathbf{x})=[1]$, is enforced with such one point quadrature.

NUMERICAL VALIDATION

The developed QC3 scheme is compared with LC1, linear finite element method (LFEM) and standard triangle quadratures with Hammer points (denoted as SH) for quadratic approximations by the following numerical examples.

Quadratic patch test

Table1 lists the displacement and energy errors in the test. The developed QC3 is the only one method which can pass the test exactly in a numerical sense while LC1 fails. Incidentally, both methods pass the linear patch test.

Table 1 Displacement and energy errors in quadratic patch test

	QC3	LC1	SH1	SH3	SH7	SH16	LFEM
E^{disp}	0.83E-12	0.13E-02	0.40E-02	0.24E-03	0.51E-04	0.25E-04	0.19E-02
E^{eng}	0.36E-11	0.99E-02	0.20E-01	0.11E-02	0.29E-03	0.11E-03	0.12E+00

Cantilever beam

Figures 1 and 2 compare the convergence and efficiency in terms of displacement and energy, respectively. The developed QC3 is obviously the most accurate and efficient method with the highest convergence rates in both displacement and energy. The LC1 is only comparable to the LFEM even if the quadratic approximation is used.

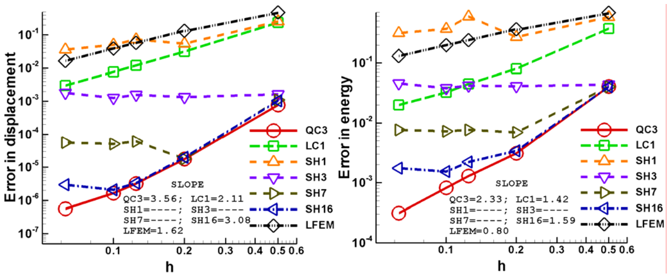

Figure 1. Convergence in displacement and energy

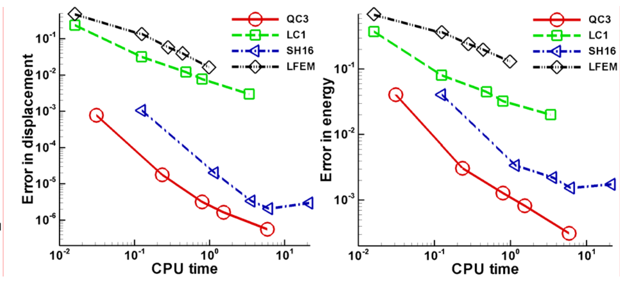

Figure 2. Efficiency in terms of displacement and energy

CONCLUSIONS

A consistency framework including the DAC and the DDC for meshfree nodal derivatives is proposed, based on which the QC3 integration scheme is developed for quadratic approximations. Due to the satisfaction of both the quadratic DAC and the quadratic DDC, QC3 passes the quadratic patch test exactly in a numerical sense and shows huge improvement for the existing LC1 and meshfree Galerkin methods with traditional triangle quadratures in convergence, accuracy and efficiency.

References

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