An advantage of spectral element method for solving incompressible Navier-Stokes equations

Zhouhua Qiu[*], Zhong Zeng[*a)], Huan Mei[*], Hiroshi Mizuseki[**], Yoshiyuki Kawazoe[**]
[*]*Department of Engineering Mechanics, Chongqing University, Chongqing 400044, China*
[**]*Institute for Materials Research, Tohoku University, Sendai 980-8577, Japan*

Summary A new Fourier-Legendre spectral element method based on the Galerkin formulation is introduced to solve the time-dependent incompressible Navier-Stokes equations in polar coordinates. The clustering of collocation points near the pole, which is happened in spectral Galerkin method, is prevented through the technique of domain decomposition in the radial direction. Therewithal the time step restriction is alleviated. A numerical example is computed and compared with the result solving by spectral Galerkin method, which reveals the advantage of spectral element method in solving the time-dependent problems.

INTRODUCTION

The approximation of the spectral method is based on the whole domain, so computation workload is too heavy for complex geometries. Actually, a main challenge for spectral methods in polar or cylindrical coordinates is the time-step restriction in time-dependent problems owing to the clustering of collocation points near the pole [2, 3]. Patera [4] first proposed the spectral element method (SEM) in 1984. The computational domain was divided into several elements, then the equations were discretized by spectral method in each element, and finally the element matrices were assembled into an overall matrix to solve the problem. We proposed a Fourier-Legendre SEM based on Galerkin variational formulation for the Poisson-type equations in polar coordinates [7]. The clustering of collocation points near the pole is avoided by the present technique of domain decomposition. With the technique, comparing to the spectral Galerkin method (SGM) [2, 3], the SEM has an advantage to alleviate time step restriction and reduce the computational burden, and this advantage is demonstrated in an example simulation.

NUMERICAL METHODS

We present the SEM in polar coordinates only. The Poisson-type equations on the unit disc in polar coordinates are

$$\begin{cases} -\nabla^2 u + \alpha u = f, \quad \Omega=\{(r,\theta): 0 \le r < 1, 0 \le \theta < 2\pi\}, \\ u|_{r=R} = g(\theta). \end{cases} \tag{1}$$

where $\nabla^2 = \partial^2/\partial r^2 + (\partial/\partial r)/r + (\partial^2/\partial\theta^2)/r^2$, α, f and g denote the given functions defined on domain Ω or its boundary Γ.
In Fourier-Legendre SEM [7], the computational region is divided into several annuluses and a small disk close to the center. In each element Ω^i, the Galerkin formulation of Eq. (1) is attained: find u, such that

$$\int_{\Omega^i} r\frac{\partial u}{\partial r}\cdot\frac{\partial \bar{v}}{\partial r}drd\theta + \int_{\Omega^i}\frac{1}{r}\frac{\partial u}{\partial\theta}\cdot\frac{\partial\bar{v}}{\partial\theta}drd\theta + \int_{\Omega^i}\alpha u\bar{v}rdrd\theta = \int_{\Omega^i} f\bar{v}rdrd\theta, \tag{2}$$

where, v is a test function, and $\bar{v}$ is its conjugation. The functions u, v, f are discretized by spectral expansion [1], the numbers of collocation points are $N_\xi+1$ and N_θ in the r and θ direction, respectively. We obtain the liner equations in Ω^i:

$$\sum_{j=0}^{N_\xi}\sum_{k=0}^{N_\theta-1} u_{jk}\left(A_{jkmn} + \alpha\cdot B_{jkmn}\right) = \sum_{j=0}^{N_\xi}\sum_{k=0}^{N_\theta-1} f_{jk}B_{jkmn}, \quad m=0,1,\cdots,N_\xi; \quad n=0,1,\cdots,N_\theta-1. \tag{3}$$

Lastly, the element stiffness matrix is obtained, then the global stiffness matrix referring to the finite element method [1], and finally the value for each node by solving the linear equations with given Dirichlet boundary conditions. The time splitting method [5] splits the Navier-Stokes equations into two Poisson-type equations, which are computed by SEM.

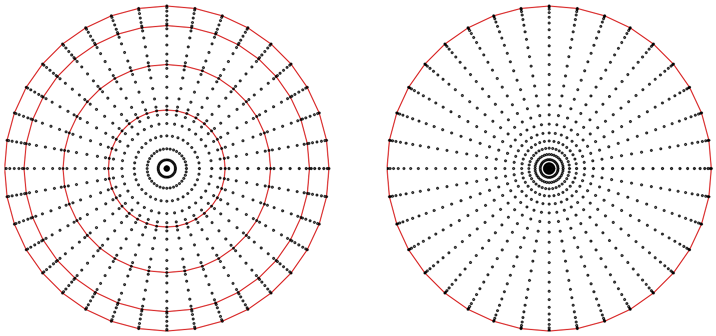
NUMERICAL RESULTS

The uniform wall driven flow in infinite cylindrical geometry, which is simplified to a 2-D problem in a disk, is considered. The analytical solution for this problem is that the azimuthal velocity in the same radial direction follows linear distribution. As shown in Fig. 1, two kinds of computational grids are used. For the first case, we make the first element boundary far from the pole with technique of domain decomposition to improve the grid quality, and the radial interval [0, 1] is divided into 4 elements, with each element consisting of 6-order quadrature points, the azimuthal direction is discretized by 30-order discrete points, and the total number of computational grid points is 30×25. For another case, the radial direction only consisted of 24-order quadrature points in the SGM and the azimuthal direction is the same as in the first case. Fig. 2 demonstrates the azimuthal velocity profile of the nodes in the direction θ=0 when the flow is stabilized with SEM and SGM. The numerical solutions are linearly distributed, which coincide with the analytical solution.

---

a) Corresponding author. Email: zzeng@cqu.edu.cn.

Then, we expound the advantage of SEM versus the SGM through the computational efficiency. Fig. 1(b) exhibits that the nodes near the origin are very close. So it will result in a very small time step by following the CFL condition when solving the time-dependent problem with an explicit scheme. It is found that the time step can be 1×10^{-5} for the SEM grids, while the maximum of only 5×10^{-7} for the SGM grids. In addition, the calculation is carried out on PC with CPU 2.20GHz, the computation of each time step is 0.018s with SEM while 0.031s with SGM because of the domain decomposition reduce the cell's aspect ratio, which leads to better-conditioned matrices and faster convergence of the matrix solvers. Moreover, the total time is about 29 minutes with SEM while 863 minutes with SGM. The numerical results indicate the advantage of SEM versus SGM in solving the time-dependent problems.



(a) (b)

Fig.1 Distribution of collocation points for SEM with 4 elements in the r direction (a) and SGM (b).

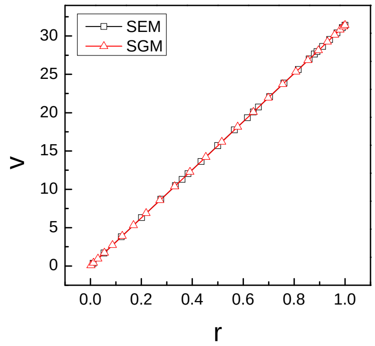


Fig. 2 The azimuthal velocity profile of the nodes in the direction θ=0 with SEM and SGM.

CONCLUTIONS

In SEM, the clustering of collocation points near the pole is prevented through the domain decomposition in the radial direction, which allows the larger time step for time-dependent problems. And the procedure is convenient. Another benefit is to reduce the cell's aspect ratio, which leads to better-conditioned matrices and faster convergence of the matrix solvers even for steady equations and unsteady implicit time schemes. Spectral element method has perfect future prospects in more fields.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the support of the National Natural Science Foundation of China (No. 10872222 and 50921063), and Research Fund for the Doctoral Program of Higher Education of China (No. 20110191110037).

References

[1] Karniadakis G.E., Sherwin S.J.: Spectral/hp Element Methods for CFD, Oxford University Press, London, 1999.
[2] Kwan Y.Y.: Efficient spectral-Galerkin methods for polar and cylindrical geometries, *Appl. Numer. Math.* 59:170–186, 2009.
[3] Shen J.: Efficient spectral-Galerkin methods III. Polar and cylindrical geometries, *SIAM J. Sci. Comput*. 18:1583–1604, 1997.
[4] Patera A.T.: A spectral element method for fluid dynamics: laminar flow in a channel expansion, *J. Comput. Phys.* 54:468–488, 1984.
[5] Karniadakis G.E., Israeli M., Orszag S.A.: High-order splitting methods for the incompressible Navier- Stokes Equations, *J. Comput. Phys.* 97:414–443, 1991.
[6] Wheeler A.A.: Four test problems for the numerical simulation of flow in Czochralski crystal growth, *J. Cryst. Growth.*102:691–695, 1990.
[7] Qiu Z.H., Zeng Z., Mei H., Li L., Yao L.P., Zhang L. Q.: A Fourier-Legendre spectral element method in polar coordinates, *J. Comput. Phys.* 231:666–675, 2012.