THE MATHEMATICA PACKAGE BVPH 1.0 FOR NONLINEAR PROBLEMS

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Summary
The Mathematica package BVPh (version 1.0) is described, together with some applications in fluid mechanics. Based on the homotopy analysis method (HAM) and computer algebra system Mathematica, the BVPh 1.0 is valid for boundary-value/eigenvalue problems governed by nonlinear ordinary differential equations with multiple solutions, singularity and multiple-point boundary conditions. The basic ideas of the HAM are briefly described, on which the BVPh 1.0 is based. A few examples are used to show that the BVPh 1.0 is a easy-to-use tool to solve nonlinear problems in mechanics.

INTRODUCTION

The world is nonlinear in essence. Like numerical techniques, analytic methods are very important tools for us to understand the essence of many nonlinear phenomenon. Among them, perturbation techniques [9] are mostly applied. By means of perturbation methods, many nonlinear problems are successfully solved and well understood. However, it is a pity that perturbation methods depend too heavily upon small physical parameter and thus is valid only for weakly nonlinear problems in general. Although some non-perturbation methods, such as Lyapunov artificial small parameter method [8], Adomian Decomposition Method [1] and so on, are proposed, all perturbation and traditional non-pertubation methods can **not** guarantee the convergence of the approximation series. So, it is valuable to develop an analytic method valid for highly nonlinear problems without any small physical parameters.

Such a kind of analytic technique was proposed first by Liao [2] in 1992 by using the basic concept "homotopy" in topology, called the homotopy analysis method (HAM) [2]-[7]. The early HAM was further modified by Liao [5] in 1997, who introduced the so-called "convergence-control parameter" for the first time to control the convergence of approximation series. As a result, the HAM is valid for highly nonlinear problems: it is the so-called "convergence-control parameter" that differs the HAM from all other analytic approximation methods. The basic ideas and typical applications are described systematically by Liao [3, 4].

BASIC IDEAS OF THE HOMOTOPY ANALYSIS METHOD

Let us consider a nonlinear differential equation $\mathcal{N}[u(x)] = 0$, where $\mathcal{N}$ is a nonlinear differential operator, $u(x)$ is a unknown function defined in a given interval $x \in \Omega$. For simplicity, we omit here all boundary/initial conditions, which can be handled in a similar way.

Let $q \in [0, 1]$ denote the embedding parameter, $u_0(x)$ the initial guess of $u(x)$, $\mathcal{L}$ an auxiliary linear operator with the property $\mathcal{L}[0] = 0$, $c_0 \neq 0$ an auxiliary parameter, called the "convergence-control parameter", respectively. Instead of solving the original nonlinear equation $\mathcal{N}[u(x)] = 0$, we construct a continuous deformation $\phi(x; q)$ with $q \in [0, 1]$ defined by the so-called zeroth-order deformation equation

$$(1 - q)\, \mathcal{L}\left[\phi(x; q) - u_0(x)\right] = c_0\, q\, \mathcal{N}[\phi(x; q)], \qquad x \in \Omega, \quad q \in [0, 1]. \tag{1}$$

Obviously, we have $\phi(x; 0) = u_0(x)$ when $q = 0$, and $\phi(x; 1) = u(x)$ when $q = 1$, respectively. Thus, as $q \in [0, 1]$ increases from 0 to 1, the solution $\phi(x; q)$ of Eq. (1) varies continuously from the initial guess $u_0(x)$ to the solution $u(x)$ of the original equation $\mathcal{N}[u(x)] = 0$. Note that Eq. (1) contains the so-called "convergence-control parameter" c_0. Besides, we have great freedom to choose $\mathcal{L}$ and $u_0(x)$. Assuming that the auxiliary linear operator $\mathcal{L}$, the initial guess $u_0(x)$, the convergence-control parameter c_0 are so properly chosen that the Maclaurin series of $\phi(x; q)$ with respect to the embedding parameter $q \in [0, 1]$, i.e.

$$\phi(x; q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x)\, q^m \tag{2}$$

is convergent at $q = 1$, we have the so-called homotopy-series

$$u(x) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x), \tag{3}$$

where $u_m(x)$ is determined by the linear mth-order deformation equation

$$\mathcal{L}\left[u_m(x) - \chi_m\, u_{m-1}(x)\right] = c_0\, \delta_{m-1}(x), \tag{4}$$

with the definition

$$\chi_k = \begin{cases} 0 & \text{when } k = 1 \\ 1 & \text{when } k \geq 2 \end{cases}, \qquad \delta_k = \frac{1}{k!}\, \frac{\partial^k \mathcal{N}[\phi(x; q)]}{\partial q^k}\bigg|_{q=0}.$$

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Here, Eq. (4) is obtained by substituting the series (2) into Eq. (1) and equating the like power of the embedding parameter q.

Since the above approach is based on the homotopy in topology, the HAM is independent of any small physical parameters at all, and besides, we have great freedom to choose the auxiliary linear operator $\mathcal{L}$, the initial guess $u_0(x)$ and the so-called "convergence-control parameter" c_0. Especially, it should be emphasized that the series (3) contains the "convergence-control parameter" c_0, which provides us a simple way to control the convergence of the approximation series (3). Thus, the HAM overcomes the restrictions of other approximation techniques. Thus, unlike perturbation and other non-perturbation techniques, the HAM is valid for highly nonlinear problems, as shown in details in Liao's book [4].

It should be emphasized that the "convergence-control" is a new concept proposed in the frame of the HAM, which differs the HAM from all other analytic approximation methods. It is found that the so-called "convergence-control parameter" c_0 provides us a simple way to guarantee the convergence of the approximation series. Using the convergence-control parameter c_0, a generalized power series about $1/(1+x)$ is deduced, which can be convergent in the whole real axis except the singular point $x = -1$, i.e. $x \in (-\infty, -1)$ and $x \in (-1, +\infty)$. Besides, in the frame of the HAM, the so-called homotopy-transform is deduced, and it is proved that the famous Euler transform is only a special case of it [4]. All of these provide us a theoretical base and reasonableness of the concept "convergence control" in the frame of the HAM.

MATHEMATICA PACKAGE BVPH 1.0 AND ITS APPLICATIONS

The HAM has been successfully applied to solve many nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs) in science, mechanics, finance and engineering. Currently, the HAM-based Mathematica package BVPh (version 1.0) is issued by Liao [4] in his book (Part II) for boundary-value/eigenvalue problems governed by one nonlinear ODE

$$\mathcal{N}[u(x)] = f(x), \qquad x \in \Omega$$

with singularity, multiple solutions and multiple-point boundary conditions, where $\mathcal{N}$ is a nonlinear differential operator, $u(x)$ is unknown function, Ω is either a finite interval $[a, b]$ or an infinite interval $[a, +\infty)$ with $a \geq 0$.

The BVPh 1.0 has been successfully applied to solve boundary-layer flows in fluid mechanics, such as Blasius flow, viscous flows over a stretching plate, and so on. For example, the multiple solutions about the viscous flow in a porous channel with orthogonally moving walls [10] can be found by means of BVPh 1.0, as shown by Liao [4]. Besides, the famous Orr-Sommerfeld eigenvalue equation is solved by the BVPh 1.0, too. It is even valid for some nonlinear PDEs related to non-similarity or unsteady boundary-layer flows. For details, please refer to Liao's book (Part II) [4].

The BVPh 1.0 is free available online (http://numericaltank.sjtu.edu.cn/BVPh.htm), together with a simple users guide and many examples. It is general and valid for many nonlinear problems in mechanics. Its new versions will be issued in future.

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