SIMULATION OF ANISOTROPIC SAVAGE-HUTTER MODEL USING GAS KINETIC SCHEME

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Summary The gas-kinetic scheme is applied to a depth-integrated continuum model for avalanche flows, namely Savage-Hutter model. In this method, the continuum fluxes are calculated based on the pseudo particle motions which are relaxed from unequilibrium to equilibrium states. The processes are described by the Bhatnagar-Gross-Krook (BGK) equation. The benefit of this scheme is its capability to resolve shock discontinuities sharply and to handle the vacuum state without special treatments. Because the Savage-Hutter equation bears an anisotropic stress on the tangential space of the topography, the equilibrium distribution function of the microscopic particles are shown to be bi-Maxwellian. These anisotropic stresses are shown the key to preserve the coordinate objectivity in the Savage-Hutter model. The effect of the anisotropic stress is illustrated by two examples: an axisymmetric dam break and a finite mass sliding on an inclined planary chute. It is found that the propagation of the flow fronts significantly depends on the orientation of the principal axes of the tangential stresses.

ANISOTROPIC SAVAGE-HUTTER MODEL AND GAS KINETIC SCHEME

Landslides, avalanches, and debris flows are gravity-driven rapid geophysical flows. Because these flows commonly exhibit the characteristics of shallowness, the shallow-water type of equations are often applied to model the phenomena. In addition, the flows contain a large portion of solid particles and present solid-like behavior. To account for the solid effects, Savage & Hutter [2] propose to use the Mohr-Coulomb soil constitutive law for the landslide or avalanche materials in the shallow-water continuum model. When the flow are in motion, the basal friction causes the materials to move at the yield condition on the basal surface. With the internal friction angle and the Mohr-Coulomb yield criterion, the stresses in the tangential plane to the basal surface are calculated and incorporated into the momentum equation.

The mass and momentum balance equations of the Savage-Hutter model for shallow, cohesionless granular flows (cf. [1]) can be written, respectively, as

$$\begin{gathered}\frac{\partial h}{\partial t} + \mathrm{div}(h\mathbf{u}) = 0,\\ \frac{\partial}{\partial t}(h\mathbf{u}) + \mathrm{div}\left(h\mathbf{u}\otimes\mathbf{u} + \frac{g_3 h^2}{2}\mathbf{K}\right) = h(g_i\mathbf{e}_i - g_3\,\mathrm{grad}\,b - \mu g_3\frac{\mathbf{u}}{|\mathbf{u}|}),\end{gathered} \tag{1}$$

where h and $\mathbf{u}$ denote the depth and velocity of the avalanche flow. The reference coordinates $\mathrm{X} = (x_1, x_2)$ are constructed to incline at the average slope of the basal surface. The basal surface is described by $b = b(\mathrm{X})$ in the x_3-direction, which is defined normal to the $\mathrm{X}-$plane and, also along this direction, the flow depth is measured. The x_3-component of the gravitational acceleration in the flow direction is g_3 and the other component of the gravitational acceleration is $\mathbf{g} = g_i\mathbf{e}_i$ in the $\mathrm{X}-$plane.

Symbol $\mathbf{K}$ denotes the matrix of the earth pressure coefficients for the tangential stresses which result from the Mohr-Coulomb soil constitutive law. $\mathbf{K}$ is a 2×2 matrix and can be diagonalized with diagonal elements, k_1 and k_2, by rotate the coordinates to the principal axes with a rotation matrix $\mathbf{T}$. Using the Mohr-Coulomb soil yield condition, these two principal pressure coefficients are functions of the internal (ϕ) and basal (δ) friction angles

$$k_1 = 2\sec^2\phi\sqrt{1-(1-\cos^2\phi/\cos^2\delta)} - 1, \quad k_2 = \frac{1}{2}\left(k_1 + 1 - \sqrt{(k_1-1)^2 + 4\tan^2\delta}\right). \tag{2}$$

The stretching (active) earth pressure coefficients are taken here for simplicity.

In this paper, we aim to extend the gas kinetic scheme, Xu [3], to the Savage-Hutter model. In this method, the flow fluxes across the interfaces between computational cells are simulated by the motions of microscopic pseudo particles. The density distribution function of the particles is assumed to follow the Bhatnagar-Gross-Krook (BGK) equation. Xu [3] demonstrate the applicability of this scheme to the hydraulic shallow-water equations. The BGK equation reads $\partial f/\partial t + \mathbf{c}\cdot\mathrm{grad}\,f = (g-f)/\tau$, where f and g are the unequilibrium and the equilibrium distribution functions, respectively. The relaxation time between the two states is denoted by τ and $\mathbf{c} = (c_1, c_2)^T$ is the velocity of the pseudo microscopic particles.

Because of the anisotropic tangential stresses of the Savage-Hutter model, the pseudo particle motions are spatially asymmetric. For a two-dimensional domain, the equilibrium distribution of the pseudo particles at any given position is a

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bi-Maxwellian distribution with its principal axes coincident with those of the Savage-Hutter equation. With this assumption, the equilibrium distribution can be written explicitly as

$$g = \frac{1}{\pi g_3 \sqrt{k_1 k_2}} \exp\left\{ -\frac{1}{g_3\, h} (\mathbf{c} - \mathbf{u})^T \ \mathbf{K}^{-1} \ \ (\mathbf{c} - \mathbf{u}) \right\}, \tag{3}$$

where $\mathbf{c}$ and $\mathbf{u}$ are the micro- and macro-velocities in the $\mathbf{X}-$coordinates. This bi-Maxwellian distribution function can be theoretically obtained by assuming the pseudo particles are distributed at the maximum entropy condition.
The macroscopic physical quantities can be constructed by taking the moment integral with respect to the column vector $\mathbf{\Theta} = (1, c_1, c_2)^T$ on the BGK equation

$$\int \mathbf{\Theta} \left(\frac{\partial f}{\partial t} + \mathbf{c} \cdot \text{grad } f \right) d\mathbf{c} = \int \mathbf{\Theta} \ \frac{g - f}{\tau} d\mathbf{c}. \tag{4}$$

When at the equilibrium limiting state, $f = g$, the above integrals yield the homogeneous equations of the Savage-Hutter model as pointed out previously. For the two-dimensional calculation, we use a dimensional splitting Godunov scheme and a Van Leer limiter for the second order reconstruction. CFL is set to 0.5 and the relaxation time is set according to Xu [3]. The details of the scheme is relegated to the designated paper.
Two simulation examples will be given in the conference but due to the space limitation of the short paper, we recapitulate the results of the sliding finite mass over an inclined planary chute. The initial mass has a semi-spherical shape at a radius $r_0 = 1.85$ (dimensionless). It is suddenly released to slide down a planary inclined chute at an inclined angle 35°. The internal and basal friction angles are 30°, giving $k_1 = 1.67$ and $k_2 = 0.45$. In the simulation, the orientation of the stress principal axes are chosen to be in parallel to the local flow velocity [4]. Fig. 1 shows the flow snapshot at $t = 5$.
The left panel of Fig. 1 indicates the orientation of the primary principal axis, counterclockwise with respect to the $x-$axis. It can be seen that the flow spreads outwards from the central line. The outlines of the anisotropic Savage-Hutter (SH) equation, the original SH, and the shallow water equation are compared in the right panel of Fig. 1. In the original SH equation, it is presumed that the primary principal axis is in the $x-$direction (for k_1) and the secondary is in y (for k_2). This principal axis orientation leads to a faster propagation flow speed in the $x-$direction than the other and results in larger flow spreading in x. On the other hand, for the shallow water equation, the flow spreads more uniformly in every direction. With the anisotropic SH equation, the flow outline lays in between the former two models. This comparison manifests that the propagation of the flow fronts significantly depends on the orientation of the principal axes of the tangential stresses.

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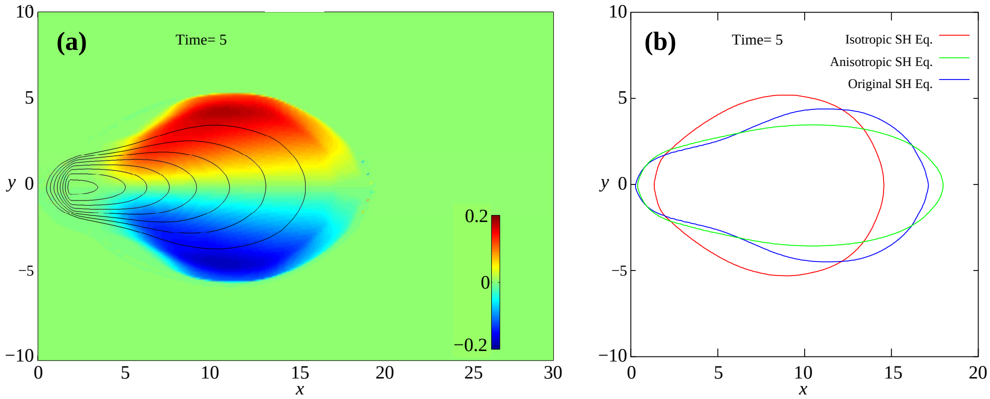

Figure 1. Finite granular mass sliding down the planary chute. (a) The orientation of the stress principal axes (b) Comparison of the outlines of the sliding mass among the shallow water (SW) equation, original coordinate dependent Savage-Hutter (SH) equation and the anisotropic SH equation.