AN ENRICHED FINITE ELEMENT FOR BIMATERIAL INTERFACIAL CRACKS

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Summary A new enriched finite element for bimaterial interfacial cracks with arbitrary crack surface tractions is presented in this paper. This element is constructed with the analytical eigensolutions and special solutions for arbitrary crack tractions, which are obtained using the symplectic approach. By using the element, interfacial crack problems can be analyzed numerically, while stress intensity factors and T-stresses can be determined directly without any extra unknowns, and transition elements or regions are also unnecessary. So dense meshes around crack tips are not required and the calculating efficiency is therefore improved. Furthermore, this new element can also be extended to crack propagation simulation.

INTRODUCTION

Stress intensity factors (SIFs) are of high interest in the study of interfacial cracks. Finite element method (FEM) is widely used in engineering to study such problems. However, conventional finite elements are too stiff to describe the nature of stress singularities near the crack tips, even though dense finite meshes are carried out around the crack tips. As a result, the simulations of structures with cracks using conventional elements are always inefficient, a lot of elements around crack tips should be carried out and the calculation quantities can hardly be reduced.
In order to overcome the shortcomings of conventional finite elements, a kind of singular finite element with stresses singularity, namely, singular elements and enriched elements were developed. Singular crack tip elements can achieve highly accuracy without extremely dense meshes around crack tips. The most popular of these elements is the quarter-point element [1, 2] which introduced a singularity of an 8-node element by moving the mid-side node on the sides connected to the crack tip to the quarter-point. Using this singular element, dense mesh around crack tips are unnecessary, but extra post processing procedure should be carried out to determine the SIFs.
Another method is enriched elements which enrich conventional elements by including the near tip fields in the trial functions [3]. The advantage of these elements is that the SIFs can be computed directly as part of the solution. But transition elements are required because the enriched elements are generally not compatible with standard elements. Another drawback of this method is that the accuracy is very sensitive to the size of the elements [4].
Recently, a lot of enriched element (or relative methods such as meshfree method (MM), XFEM, etc) were reported in which, however, the shortcomings still exist, namely, SIFs cannot be obtained directly in singular elements, extra unknowns and transition regions are still required around enriched elements [5].
In this study, a new enriched finite element is developed to study interfacial cracks with arbitrary crack surface tractions. The novel element is constructed with the analytical solution obtained using the symplectic dual approach. So SIFs can be determined directly without any extra unknowns, besides, transition elements are not required.

Analytical solution and variational principle

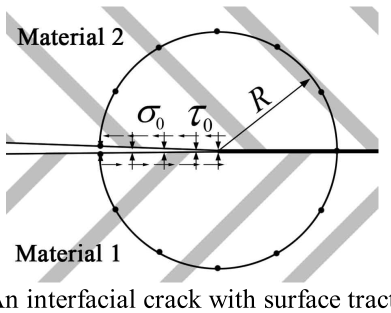

Fig 1. An interfacial crack with surface traction

Using the symplectic approach, variables are arranged in a dual vector $\boldsymbol{Z}=[u,v,S_r,S_{r\theta}]^{\mathrm{T}}$ in which u,v are displacements while $S_r=r\sigma_r$, $S_{r\theta}=r\tau_{r\theta}$ are stress components. $\xi=\ln r$ is introduced to simplify mathematical derivation.
So the general solution of a bimaterial crack problem can be obtained as

$$\boldsymbol{Z}(\xi,\theta)=[u,v,S_r,S_{r\theta}]^{\mathrm{T}}=\tilde{\boldsymbol{Z}}(\xi,\theta)+\hat{\boldsymbol{Z}}(\xi,\theta)=\tilde{\boldsymbol{Z}}(\xi,\theta)+\sum_{n=1}^{\infty}[a_n\exp(\mu_n\xi)\boldsymbol{\psi}_n(\theta)] \tag{1}$$

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where $\tilde{Z}$ is the special solution for arbitrary tractions, $\psi_n(\theta)$ is the eigensolution corresponding to symplectic eigenvalue μ_n, and $a_n (n=1,2,\cdots)$ are undetermined coefficients.
In symplectic space, $\psi_n(\theta)$ can be obtained by solving following equation

$$\boldsymbol{H}\psi(\theta) = \mu\psi(\theta) \tag{2}$$

where $\boldsymbol{H}$ is an operator matrix. The details are referred to [6]. Actually, this analytical solution $\hat{\boldsymbol{Z}}(\xi,\theta)$ is the same as the well known Williams solution. But the benefit of using the symplectic approach is the special solution $\tilde{\boldsymbol{Z}}$ can be solved analytically. The expressions of $\tilde{\boldsymbol{Z}}$ are not listed there.

Discrete equation

Another benefit of using the symplectic approach is that deformation energy of the enriched element can be obtained [7]

$$U = \iint_{\Omega} \left(uS_r + vS_{r\theta} \right) \mathrm{d}\Omega \tag{3}$$

If the element is circle with radius R, Eq. (3) is simplified into:

$$U = \int_{-\pi}^{\pi} \left(uS_r + vS_{r\theta} \right)_{\xi=\ln R} \mathrm{d}\theta \tag{4}$$

Noted here that the variables in Eq. (3) and (4) are analytical solutions, otherwise they cannot be true if any other trail functions are used. Substituting the analytical solution into (4) gives

$$U = \left(\frac{1}{2} \boldsymbol{a}^T \cdot \boldsymbol{R}_a \cdot \boldsymbol{a} + \boldsymbol{a}^T \cdot \boldsymbol{F}_a \right) + U^* \tag{5}$$

$$\boldsymbol{R}_a = \int_{-\pi}^{\pi} (\hat{S}_r \hat{u} + \hat{S}_{r\theta} \hat{v}) \Big|_{\xi=\ln R} \mathrm{d}\theta , \quad \boldsymbol{F}_a = \int_{-\pi}^{\pi} (\tilde{S}_r \hat{u} + \tilde{S}_{r\theta} \hat{v}) \Big|_{\xi=\ln R} \mathrm{d}\theta \tag{6}$$

where U^* is a constant. $\boldsymbol{F}_a = \boldsymbol{0}$ when a crack is traction free. Consider a circle finite element as shown in Fig. 1, according to Eq. (1), displacements on export nodes $\boldsymbol{d}$ can be expressed as:

$$\boldsymbol{d} - \tilde{\boldsymbol{d}} = \boldsymbol{T} \cdot \boldsymbol{a} \;, \quad \boldsymbol{a} = \boldsymbol{T}^{-1} \cdot (\boldsymbol{d} - \tilde{\boldsymbol{d}}) \tag{7}$$

Replace the unknowns $\boldsymbol{a}$ in (5) with $\boldsymbol{d}$ gives:

$$U = \frac{1}{2} \boldsymbol{d}^{\mathrm{T}} \cdot \boldsymbol{R}_e \cdot \boldsymbol{d} - \boldsymbol{d}^{\mathrm{T}} \cdot \boldsymbol{F}_e + U^* \tag{8}$$

Finally, equivalent stiffness matrix $\boldsymbol{R}_e$ and equivalent load vector $\boldsymbol{F}_e$ of the enriched element is given.
As the nodal value of this new element is nodal displacements, no transition element or regions are required. Assemble the enriched element into a conventional FEM system, the global stiffness matrix and nodal force vector are specified:

$$\boldsymbol{KU} = \boldsymbol{P} \tag{9}$$

Once the nodal displacements are solved, SIFs can be solved directly via Eq. (7) in which the expanding constant associate with the eigenvalue $0 < \mu < 1$ are the SIFs, and the $\mu = 1$ term is the T-stress. Besides, higher expanding terms can also be determined directly accordingly.

CONCLUSIONS

An enriched finite element constructed with the analytical solution is presented. It can be applied to analyze the mixed-mode interfacial cracks numerically. Dense meshes are not required around crack tips, and the calculative efficiency is therefore improved. Besides, neither transition elements nor extra unknowns are required. The present method can be applied to crack propagations, and can be coupled with other numerical methods.

References

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