A SEPARATION-VARIABLE QUADRATURE ELEMENT METHOD FOR THICKNESS-SHEAR VIBRATIONS OF RECTANGULAR CRYSTAL PLATES

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Summary This paper presents a separation-variable quadrature element method for thickness-shear vibrations of rectangular crystal plates based on the differential quadrature finite element method (DQFEM) and the Kantorovich technique. The total potential functional of the two-dimensional elasticity problem is iteratively reduced to one dimensional and then solved by DQFEM. It is shown that the fundamental thickness-shear frequency can be obtained quite efficiently and accurately.

INTRODUCTION

As the core issue of a quartz crystal resonator, vibration analysis of a crystal plate, usually circular or rectangular plate with free edges, is always required for the essential information regarding properties such as the vibration frequency and mode coupling. The vibration analysis of a crystal plate in the thickness-shear mode, which is one of important functioning modes of quartz crystal resonators, must be done accurately to determine design parameters which will reduce the coupling with spurious modes and enhance the resonance through the quality factor [1, 2]. However, exact solutions could be obtained only for rectangular crystal plates with a pair of opposite edges simply supported [1] or under some restrictions on length/width and Poisson's ratios [3]. The computational cost is very expensive if a numerical procedure is used [2]. In this paper, the total potential functional for crystal plates is first reduced to one-dimension by the Kantorovich technique then solved by the differential quadrature finite element method (DQFEM) [4] with high accuracy. Then the solution obtained by DQFEM is used to reduce the functional again. Using the same iteration approach, it is shown that convergent thickness-shear frequency can be obtained after three to five iterations usually. This technique is named as separation-variable quadrature element method (SVQEM). The main advantage of the SVQEM is that the two dimensional problem is reduced to one dimension and could be solved quite accurately and efficiently.

THE SVQEM FOR CRYSTAL PLATES

The total potential functional

Consider a rectangular crystal plate of length $2a$, width $2b$ and uniform thickness h. The plate is discretised as m strips (elements) in x-direction because the fundamental thickness-shear mode is nearly constant with respect to y. The total potential functional for each element of the plate is

$$\begin{aligned} U^s &= \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a}\left[D_1\left(\frac{\partial\phi}{\partial x}\right)^2 + 2D_2\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y}\right) + D_3\left(\frac{\partial\psi}{\partial y}\right)^2 + D_5\left(\frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial x}\right)^2\right]\mathrm{d}x\mathrm{d}y \\ &+ \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a}\left[D_6\left(\frac{\partial w}{\partial x}-\phi\right)^2 + D_4\left(\frac{\partial w}{\partial y}-\psi\right)^2\right]\mathrm{d}x\mathrm{d}y - \frac{\omega^2\rho}{2}\int_{-b}^{b}\int_{-a}^{a}\left(hw^2+J\phi^2+J\psi^2\right)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{1}$$

where $J = h^3/12$, w is the deflection, and ϕ and ψ are the angles of rotations of a normal line due to plate bending with respect to x and y coordinates, respectively, and

$$\begin{gathered} D_1 = \frac{h^3}{12}\left(C_{11}-\frac{C_{12}^2}{C_{22}}\right),\quad D_2 = \frac{h^3}{12}\left(C_{13}-\frac{C_{12}C_{23}}{C_{22}}\right),\quad D_3 = \frac{h^3}{12}\left(C_{33}-\frac{C_{23}^2}{C_{22}}\right) \\ D_4 = \frac{\pi^2 h}{12}\left(C_{44}-\frac{C_{24}^2}{C_{22}}\right),\quad D_5 = \frac{h^3}{12}C_{55},\quad D_6 = \frac{\pi^2 h}{12}C_{66} \end{gathered} \tag{2}$$

in which C_{ij} are elastic constants referred to x, y, z axes fixed in the plate. (Note that D_4 and D_6 have involved the shear effects). We need to solve the fundamental thickness-shear frequency of magnitude

$$\omega_0 = \frac{\pi}{h}\sqrt{\frac{C_{66}}{\rho}} \tag{3}$$

The differential quadrature finite element method (DQFEM)

The DQFEM [4] will be used to discretize the total potential functional. Thus, two fundamental equations of the DQFEM are presented here. The nth derivatives of the field variable $f(x)$ at point x_i is approximated by the DQM as

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$$f_i^{(n)} = \sum_{j=1}^{N} A_{ij}^{(n)} f_j \quad \left(i = 1, 2, \cdots, N\right) \tag{4}$$

where $A_{ij}^{(n)}$ are the weighting coefficients of the nth order derivatives, and N the number of grid points in the x-direction. The Gauss-Lobatto quadrature rule with precision degree ($2N-3$) for function $f(x)$ defined at $[-1, 1]$ is

$$\int_{-1}^{1} f\left(x\right) \mathrm{d}x = \sum_{j=1}^{N} C_j f\left(x_j\right) \tag{5}$$

where C_j are the Gauss-Lobatto integration weights.

The SVQEM for crystal plates

According to the Kantorovich method, the deflection w, and the angles of rotations ϕ and ψ can be separated into two uncoupled functions as:

$$w = w_x\left(x\right) w_y\left(y\right), \quad \phi = \phi_x\left(x\right)\phi_y\left(y\right), \quad \psi = \psi_x\left(x\right)\psi_y\left(y\right) \tag{6}$$

Substituting expressions (6) into Eq.(1) and using the DQFEM [4] one can obtain

$$U^s = Q_2 + Q_3 + Q_4 - \rho\omega^2 Q_1 \tag{7}$$

where

$$\begin{aligned}
Q_1 &= hS_3T_3 + J\left(S_1T_1 + S_2T_2\right) \\
Q_2 &= D_1S_1T_7 + 2D_2S_4T_4 + D_3S_8T_2 + D_5\left(S_7T_1 + S_2T_8 + 2S_5T_5\right) \\
Q_3 &= D_6\left(S_3T_{10} - 2S_6T_6 + S_1T_1\right) \\
Q_4 &= D_4\left(S_{10}T_3 - 2S_9T_9 + S_2T_2\right)
\end{aligned} \tag{8}$$

in which

$$\begin{aligned}
&T_1 = \boldsymbol{\phi}_x^{\mathrm{T}}\boldsymbol{C}\boldsymbol{\phi}_x, \quad T_2 = \boldsymbol{\psi}_x^{\mathrm{T}}\boldsymbol{C}\boldsymbol{\psi}_x, \quad T_3 = \boldsymbol{w}_x^{\mathrm{T}}\boldsymbol{C}\boldsymbol{w}_x, \quad T_4 = \boldsymbol{\phi}_x^{\mathrm{T}}\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{\psi}_x, \quad T_5 = \boldsymbol{\psi}_x^{\mathrm{T}}\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{\phi}_x, \quad T_6 = \boldsymbol{w}_x^{\mathrm{T}}\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{\phi}_x \\
&T_7 = \boldsymbol{\phi}_x^{\mathrm{T}}\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{A}^{(1)}\boldsymbol{\phi}_x, \quad T_8 = \boldsymbol{\psi}_x^{\mathrm{T}}\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{A}^{(1)}\boldsymbol{\psi}_x, \quad T_9 = \boldsymbol{w}_x^{\mathrm{T}}\boldsymbol{C}\boldsymbol{\psi}_x, \quad T_{10} = \boldsymbol{w}_x^{\mathrm{T}}\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{A}^{(1)}\boldsymbol{w}_x
\end{aligned} \tag{9}$$

where $\boldsymbol{C}$ = diag$[C_k]$, k=1, 2,…, N, and

$$\boldsymbol{\phi}_x^{\mathrm{T}} = \left[\phi_{x1} \quad \phi_{x2} \quad \cdots \quad \phi_{xN}\right], \quad \boldsymbol{\psi}_x^{\mathrm{T}} = \left[\psi_{x1} \quad \psi_{x2} \quad \cdots \quad \psi_{xN}\right], \quad \boldsymbol{w}_x^{\mathrm{T}} = \left[w_{x1} \quad w_{x2} \quad \cdots \quad w_{xN}\right] \tag{10}$$

S_j (j=1, 2,…,10) are the functions of y and similar as Eqs.(9). First, assume w_y, ϕ_y, ψ_y (or w_x, ϕ_x, ψ_x) as known functions. For fundamental thickness-shear mode, w_y, ϕ_y, ψ_y are nearly constants. Applying the variational principle, one obtains the mass and stiffness matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ and the displacement vector $\boldsymbol{U}_x^s$ of each strip as

$$\boldsymbol{K} = \begin{bmatrix} D_1S_1\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{A}^{(1)} + \left(D_5S_7 + D_6S_1\right)\boldsymbol{C} & D_2S_4\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C} + D_5S_5\boldsymbol{C}\boldsymbol{A}^{(1)} & -D_6S_6\boldsymbol{C}\boldsymbol{A}^{(1)} \\ D_2S_4\boldsymbol{C}\boldsymbol{A}^{(1)} + D_5S_5\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C} & D_5S_2\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{A}^{(1)} + \left(D_3S_8 + D_4S_2\right)\boldsymbol{C} & -D_4S_9\boldsymbol{C} \\ -D_6S_6\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C} & -D_4S_9\boldsymbol{C} & D_6S_3\boldsymbol{A}^{(1)\mathrm{T}}\boldsymbol{C}\boldsymbol{A}^{(1)} + D_4S_{10}\boldsymbol{C} \end{bmatrix} \tag{11}$$

$$\boldsymbol{M} = \rho\begin{bmatrix} JS_1\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & JS_2\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & hS_3\boldsymbol{C} \end{bmatrix}, \quad \boldsymbol{U}_x^s = \begin{bmatrix} \boldsymbol{\phi}_x \\ \boldsymbol{\psi}_x \\ \boldsymbol{w}_x \end{bmatrix} \tag{12}$$

Thus, the crystal plate is reduced as a one dimensional beam structure. After the Fundamental Thickness-shear Mode and Frequency (FTMF) are obtained, the discrete values of $\boldsymbol{w}_x$, $\boldsymbol{\phi}_x$, $\boldsymbol{\psi}_x$ are substituted into Eqs.(7) and (8) to solve new $\boldsymbol{w}_y$, $\boldsymbol{\phi}_y$, $\boldsymbol{\psi}_y$ of the FTMF for each strip. Using the same iteration approach, it is shown that convergent thickness-shear frequency can be obtained after several iterations usually. It may be seen that the SVQEM reduced computational cost $m\times N^2$ (if FEM is used) to no more than five times of $m\times N$, which is quite promising especially when N is large.

CONCLUSIONS

The separation-variable quadrature element method (SVQEM) incorporated the Kantorovich technique (or the separation-variable method) into the differential quadrature finite element method (DQFEM). The SVQEM reduced computational cost $m\times N^2$ (if FEM is used) to no more than five times of $m\times N$, is quite efficient, and therefore, is capable of solving problems more accurately. It may be understood that SVQEM is a general method for elasticity. Two dimensional elasticity solutions can be employed to obtain three dimensional solutions through this technique.

References

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