An Improved Lattice Boltzmann Model for Incompressible Flow

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Summary Based on the Hermite expansion technique, an improved incompressible lattice Boltzmann model is proposed. Using the model, the incompressible Navier-Stokes equations can be derived from both the Boltzmann-BGK equation and the lattice Boltzmann equation. Besides, a new formula for pressure is designed. To validate the model, the lid-driven cavity flow is simulated, and the results domonstrate that the new model has better accuracy than the D2G9 model in the region of high deviatoric stress.

INTRODUCTION

In 2000, Guo proposed the D2G9 model [1], which has been widely applied to incompressible flow. Using Guo's incompressible model, the incompressible Navier-Stokes equations can be recovered under incompressible conditions [2]. Based on the D2G9 model, an improved incompressible model is proposed in this paper. Firstly, the fluid density which is 0 in Guo's model is set to be 1 in present model. Secondly, we modify the moment equations of the density distribution function and its equilibrium form, and then the equilibrium density distribution function is derived through the Hermite expansion [3]. And finally, the equilibrium density distribution function is discretized by applying the discrete velocity of the D2Q9 model [4, 5], and the formula for pressure is proposed according to the second-order moment equation of the distribution function. The LBM code based on the present model is validated based on the numerical simulations of cavity flow.

MATHEMATICAL FORMULATIONS

In this section, we introduce the improved incompressible model. The evolution equation of the density distribution function is:

$$f_\alpha(\mathbf{X}+\xi_\alpha dt, t+dt) - f_\alpha(\mathbf{X},t) = -\frac{1}{\tau}[f_\alpha(\mathbf{X},t) - f_\alpha^{(0)}(\mathbf{X},t)] \tag{1}$$

where f_α and $f_\alpha^{(0)}$ indicate the distribution function and its equilibrium form respectively, τ is the dimensionless relaxation time. The modified moment equations of the present model are

$$\int f d\xi = \int f^{(eq)} d\xi = 1 \tag{2}$$

$$\int f\xi d\xi = \int f^{(eq)}\xi d\xi = \mathbf{u} \tag{3}$$

$$\int f\xi_i\xi_j d\xi = u_i u_j - P_{ij} \tag{4}$$

$$\int f^{(eq)}\xi_i\xi_j d\xi = u_i u_j + p\delta_{ij} \tag{5}$$

where P_{ij} is the stress tensor, p is the fluid pressure, Based on the modified moment equations of the equilibrium density distribution function, the continuous expression of the 2D equilibrium density distribution function is obtained as

$$f^{(eq)} = \frac{1}{(2\pi RT)^{D/2}}\exp(-\frac{\xi^2}{2RT})\left\{1+\frac{\xi\bullet\mathbf{u}}{RT}+\frac{(\xi\bullet\mathbf{u})^2}{2(RT)^2}-\frac{\mathbf{u}^2}{2RT}+(\frac{p}{RT}-1)(\frac{\xi^2}{2RT}-1)\right\} \tag{6}$$

According to the discrete velocity of the D2Q9 model, the discretization in momentum space is accomplished as

$$s_\alpha(\mathbf{u}) = w_\alpha[\frac{3(\xi_\alpha\bullet\mathbf{u})}{c^2}+\frac{9(\xi_\alpha\bullet\mathbf{u})^2}{2c^4}-\frac{3\mathbf{u}^2}{2c^2}] \tag{7}$$

where c=dx/dt, dx, dt are the spatial and time step, respectively. The expression of $s_\alpha(\mathbf{u})$ is

$$f_\alpha^{(0)}(\mathbf{X},t) = \begin{cases} 8/9 - 4p/3c^2 + s_\alpha(\mathbf{u}) & \alpha = 0 \\ 1/18 + p/6c^2 + s_\alpha(\mathbf{u}) & \alpha = 1,2,3,4 \\ -1/36 + p/6c^2 + s_\alpha(\mathbf{u}) & \alpha = 5,6,7,8 \end{cases} \tag{8}$$

where $w_0 = 4/9$, $w_{1,2,3,4} = 1/9$, $w_{5,6,7,8} = 1/36$. The macroscopic fluid velocity is calculated by

$$\mathbf{u} = \sum_{\alpha=1}^{8} f_\alpha\xi_\alpha \tag{9}$$

Based on the second-order moment equation (4), we proposed a new formula for the fluid pressure

$$p = (\sum_{\alpha=1}^{8} f_\alpha\xi_{\alpha i}\xi_{\alpha i} - u_i u_i)/2 \tag{10}$$

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Applying the above –mentioned incompressible LBM model, the incompressible N-S equations can be obtained as

$$\nabla \bullet \mathbf{u} = 0 \tag{11}$$

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \bullet (\mathbf{uu}) = -\nabla p + \nu \Delta \mathbf{u} \tag{12}$$

RESULTS

In this section, we present the numerical results of the lid-driven cavity flow with Re=1000. The results of present model are in good agreement with those produced by other methods [1, 6]. Besides, the comparisons with the results of Guo's incompressible model demonstrate that the present model is more accurate than the D2G9 model in the region of high deviatoric stress.

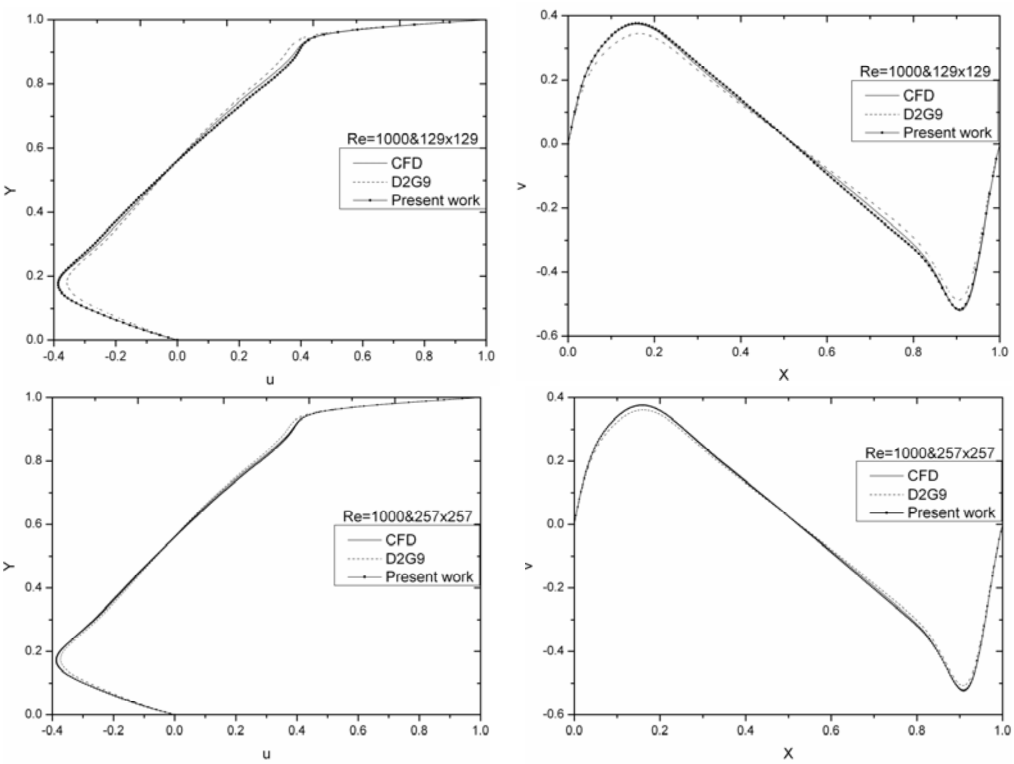

FIG.1 the velocity components, u and v, along the vertical and horizontal centerlines at Re=1000.

CONCLUSIONS

In this paper, on the basis of Guo's D2G9 model, we propose an improved lattice Boltzmann model for incompressible flow. The present model inherits the advantages of the D2G9 model in which the density is a constant and the fluid pressure is independent from density. In addition, with the new formula of fluid pressure, the present model takes the viscous effects of the fluid flow into account with higher accuracy, which is demonstrated by the numerical simulation for the lid-driven cavity flow.

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