Natural mode analysis of elliptical cylindrical cavities with multiple elliptical cylinders by using the collocation multipole method

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Summary The eigenproblem is solved by the collocation multipole method. The multipole expansion is formulated in terms of Mathieu functions to deal with elliptical boundaries. The direct searching approach is employed to determine the natural frequencies by using the singular value decomposition (SVD). The accuracy and numerical convergence is validated by comparison with the commercial finite-element code ABAQUS. Excellent accuracy and fast rate of convergence are the key features thanks to its semi-analytical feature.

INTRODUCTION

The natural mode analysis of an acoustic cavity can provide important information in the design phase for the cavity related to mechanical systems such as automobile mufflers and hermetic compressors. Although numerical methods such as finite element methods (FEM) and boundary element methods (BEM) can solve these problems, analytical solutions, if available, usually result in accurate and fast-rate convergence and provide physical insight for the problem being considered. Consequently, a semi-analytical approach to the eigenproblem for a 3D acoustic cavity with elliptical boundaries is presented in this paper.

The concept of the multipole method to solve multiply connected domain problems was firstly devised by Zȧvis̆ka and used for the interaction of waves with arrays of circular cylinders by Linton and Evans. The addition theorem is often employed to transform the multipole expansion into one of coordinate systems for satisfying the specified boundary conditions. For the circular boundary, some applications can be seen in the literature, including the water wave scattering problem, the free vibration problem of circular membranes, the free vibration problem of circular plates and the flexural wave scattering problem. Furthermore, Chatjigeorgiou et al. proposed an analytical approach to the hydrodynamic diffraction by multiple elliptical cylinders using the addition theorem of the Mathieu functions. In the view point of mathematics, the procedure is exact and elegant. But we need to face the complicated formulation and the associated numerical calculation due to its complicated form for the addition theorem. Moreover, the usage of the addition theorem limits its application to the problem with canonical boundaries such as a circle or ellipse.

This paper presents a collocation multipole method to solve the eigenproblem of a three-dimensional acoustic cavity with a multiply connected domain. Instead of using the addition theorem, the multipole method will be extended to the problems with elliptic boundaries by uniformly collocating points on the boundaries to satisfy the boundary conditions. The normal derivatives at the collocation points are exactly calculated by using the directional derivative in each coordinate system. In this way, a coupled finite linear algebraic system is derived. For the eigenproblem, based on the direct searching approach, the nontrivial eigenvalues can be found by detecting the zero determinant of the linear system through the technique of singular value decomposition (SVD). The solution accuracy and computing efficiency of the proposed results are to be compared critically with those of available analytic solutions and such numerical method as the FEM

THE HELMHOLTZ EQUATION AND ITS GENERAL SOLUTION IN THE ELLIPTIC CYLINDRICAL COORDINATE SYSTEM

In the elliptic cylindrical coordinates, the Helmholtz equation has separated solutions of the form

$$\mathrm{Mc}_m^{(i)}(\xi,q)\mathrm{ce}_m(\eta,q)Z_l(\mathrm{z}) \quad \text{and} \quad \mathrm{Ms}_m^{(i)}(\xi,q)\mathrm{se}_m(\eta,q)Z_l(\mathrm{z}), \quad i=1,\ 2,\ 3 \text{ and } 4.$$

where ce_m and se_m are the even and the odd angular Mathieu functions, respectively, $\mathrm{Mc}_m^{(i)}$ and $\mathrm{Ms}_m^{(i)}$ are the even and the odd radial Mathieu functions, respectively.

Since the product $\mathrm{Mc}_m^{(2)}(\xi,q)\mathrm{ce}_m(\eta,q)$ dose not satisfy the continuity in gradient across the interfocal line and the product $\mathrm{Ms}_m^{(2)}(\xi,q)\mathrm{se}_m(\eta,q)$ dose not satisfy the function continuity [1], the permissible solution is

$$p^I(\xi,\eta,z)=\left[\sum_{m=0}^{\infty}c_m\,\mathrm{Mc}_m^{(1)}(\xi,q)\mathrm{ce}_m(\eta,q)+\sum_{m=1}^{\infty}s_m\,\mathrm{Ms}_m^{(1)}(\xi,q)\mathrm{se}_m(\eta,q)\right]Z_l(z), \tag{1}$$

for the interior domain to the elliptical boundary, where l=0, 1, 2, … and the coefficients c_{mn} and s_{mn} are to be determined by the boundary conditions, and

$$p^E(\xi,\eta,z)=\left[\sum_{m=0}^{\infty}c_m\,\mathrm{Mc}_m^{(3)}(\xi,q)\mathrm{ce}_m(\eta,q)+\sum_{m=1}^{\infty}s_m\,\mathrm{Ms}_m^{(3)}(\xi,q)\mathrm{se}_m(\eta,q)\right]Z_l(z), \tag{2}$$

for the exterior domain.

THE MULTIPOLE METHOD

The problem of an elliptical cavity with H elliptical cylinders can be decomposed into one interior domain problem and H exterior domain problems. For this multiply-connected domain problem, the acoustic pressure can be explicitly expressed as an infinite sum of multipoles at the center of each ellipse,

$$\begin{aligned} P(\boldsymbol{x}) = \Bigg\{ & c_0^1 \,\mathrm{Mc}_0^{(1)}(\xi_1,q)\mathrm{ce}_0(\eta_1,q) + \sum_{m=1}^{\infty} c_m^1 \,\mathrm{Mc}_m^{(1)}(\xi_1,q)\mathrm{ce}_m(\eta_1,q) + s_m^1 \,\mathrm{Ms}_m^{(1)}(\xi_1,q)\mathrm{se}_m(\eta_1,q) \\ & + \sum_{j=2}^{L}\left[c_0^j \,\mathrm{Mc}_0^{(3)}(\xi_j,q)\mathrm{ce}_0(\eta_j,q) + \sum_{m=1}^{\infty} c_m^j \,\mathrm{Mc}_m^{(3)}(\xi_j,q)\mathrm{ce}_m(\eta_j,q) + s_m^j \,\mathrm{Ms}_m^{(3)}(\xi_j,q)\mathrm{se}_m(\eta_j,q)\right]\Bigg\} Z_l(z), \end{aligned} \tag{3}$$

THE COLLOCATION MULTIPOLE METHOD

In order to avoid the complex application of the addition theorem, the collocation method is adopted to calculate Eq. (3) directly in each local elliptic cylindrical coordinate system. By uniformly collocating N (=2M+1) points on each elliptical boundary and truncating the infinite series, we have.

$$\begin{aligned} \frac{\partial}{\partial n}P(\boldsymbol{x}) = \Bigg\{ & c_0^1 \frac{\partial}{\partial n}\left(\mathrm{Mc}_0^{(1)}(\xi_1^{gn},q)\mathrm{ce}_0(\eta_1^{gn},q)\right) + \sum_{m=1}^{M} c_m^1 \frac{\partial}{\partial n}\left(\mathrm{Mc}_m^{(1)}(\xi_1^{gn},q)\mathrm{ce}_m(\eta_1^{gn},q)\right) + s_m^1 \frac{\partial}{\partial n}\left(\mathrm{Ms}_m^{(1)}(\xi_1^{gn},q)\mathrm{se}_m(\eta_1^{gn},q)\right) + \\ & \sum_{j=2}^{L}\left[c_0^j \frac{\partial}{\partial n}\left(\mathrm{Mc}_0^{(3)}(\xi_j^{gn},q)\mathrm{ce}_0(\eta_j^{gn},q)\right) + \sum_{m=1}^{M} c_m^j \frac{\partial}{\partial n}\left(\mathrm{Mc}_m^{(3)}(\xi_j^{gn},q)\mathrm{ce}_m(\eta_j^{gn},q)\right) + s_m^j \frac{\partial}{\partial n}\left(\mathrm{Ms}_m^{(3)}(\xi_j^{gn},q)\mathrm{se}_m(\eta_j^{gn},q)\right)\right]\Bigg\} Z_l(z) \end{aligned} \tag{4}$$

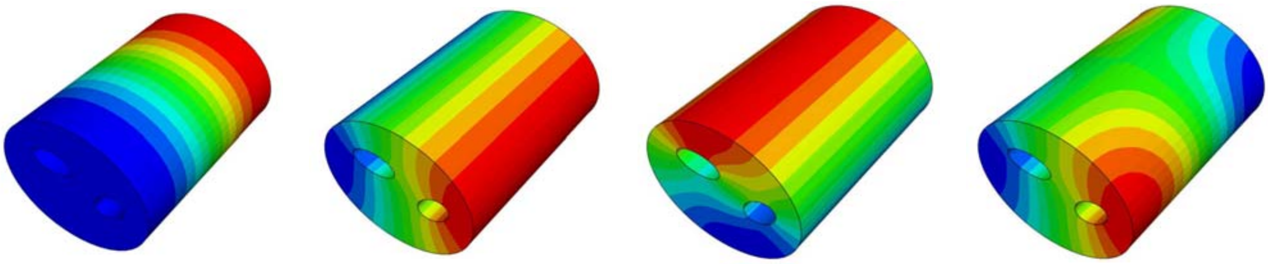
for g=1, 2, …, Q and n=1, 2, 3, …, N, where ξ_j^{gn} and η_j^{gn} [2] denote elliptical coordinates of the nth collocation point on the gth elliptical boundary with respect to the jth local elliptical coordinate system.

RESULTS AND CONCLUSIONS

Table Ⅰ summarizes the lower six natural frequencies for an elliptical cylindrical cavity with two elliptical cylinders by using the present method and the FEM. In comparison to the results provided by various finite element meshes, the present results show excellent accuracy. Figure 1 shows the lower fours natural modes with natural frequencies. It follows from several numerical experiments that the proposed formulation does not yield spurious eigenvalues. Numerical results demonstrate that the present method has good accuracy and fast rate of convergence.

Table Ⅰ The lower six natural frequencies for an elliptical cylindrical cavity with two elliptical cylinders by using the present method and the FEM

Natural frequency	Present method	FEM(ABAQUS)		
		16759 nodes	5672 nodes	2466 nodes
k_1	7.8540	7.854	7.8541	7.8544
k_2	12.612	12.614	12.615	12.619
k_3	13.674	13.678	13.680	13.694
k_4	14.857	14.859	14.860	14.863
k_5	15.708	15.710	15.712	15.720
k_6	15.769	15.772	15.774	15.786



(a) k=7.8540 (b) k=12.612 (c) k=13.674 (d) k=14.857

Fig. 1 The lower fours natural modes with natural frequencies for an elliptical cylindrical cavity with two elliptical cylinders.

References

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[2] W.M. Lee, 2011, Natural mode analysis of an acoustic cavity with multiple elliptical boundaries by using the collocation multipole method, Journal of Sound and Vibration, Vol. 330, Issue 20, pp. 4915-4929.