THE ANALYTIC STUDY OF ELASTIC METAMATERIALS WITH NEGATIVE MASS AND MODULUS

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Summary Compared with the study of electromagnetic metamaterials, much less attention has been paid to elastic metamaterials. Some elastic metamaterial models have been developed and used to simulate the single-negative or double negative properties at certain frequencies. In the current study a new model is proposed, which contains rotational elements, to generate both negative mass and modulus in a controlled manner. The mechanisms behind the negative mass and modulus are discussed and the wave propagation in this material system is investigated.

INTRODUCTION

Significant progress in electromagnetic metamaterials has been achieved by realizing negative electric permittivity and magnetic permeability[1]. In comparison, the study of elastic/acoustic metamaterials is still very limited, although some progress has been made[2]. Different simplified models have been developed and used to simulate single-negative or double negative elastic/acoustic metamaterials at certain frequencies[3,4]. Negative mass can be achieved using these models but negative modulus cannot usually be fully controlled. In the current study the generation of controllable negative mass and modulus is studied theoretically based on a spring-mass-type model containing rotational motion. The results indicate that by adjusting the geometry and material properties of the system, double negative metamaterials (negative mass and modulus) can be reliably achieved in a controlled manner. The resulting wave propagation and dispersion property are discussed. The current model can be easily extended to two-dimensional cases.

FORMULATION OF THE PROBLEM

The problem considered is the harmonic response of a one-dimensional elastic metamaterial. The proposed model is a periodic structure with its unit cell being given in Fig. 1, in which a rigid body with mass m and moment of inertia I is attached to four linear springs with elastic constants k and k', which are connected to two other rigid bodies with mass M. The length of the unit is assumed to be L. The mass centers are assumed to be at the centers of the rigid bodies. The motion of the unit is limited to only the longitudinal direction (horizontal) and the transverse motion of the system will not be considered. It should be mentioned that the asymmetry in the transverse direction, caused by the arrangement of the springs will not contribute to the horizontal motion. The rotating motion of masses at the two ends will be ignored, which can be easily eliminated by superimposing a second cell rotated 180 degrees about the horizontal axis.

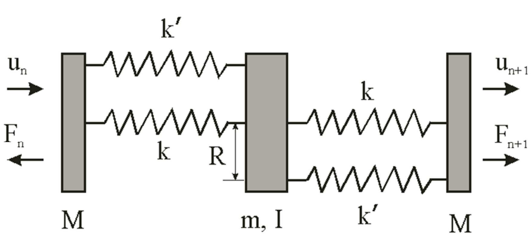

Fig. 1 *A unit cell of the metamaterial model*

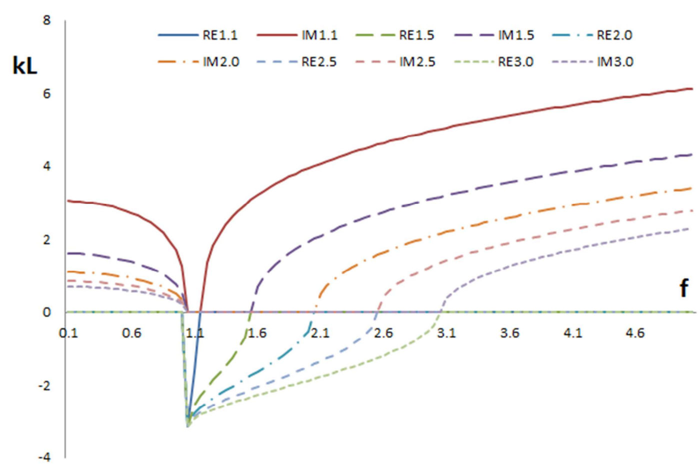

Fig. 2 *Dispersion curve: the wave number for normalized frequency $f=\omega/\omega_1$; the legends indicating real(RE) or imaginary(IM) with the numbers being the ratio ω_0/ω_1*

Double negative behaviour

The system is subjected to a harmonic excitation of frequency ω. For convenience only the amplitude of the resulting displacement and force will be considered in the following discussion. The amplitudes of displacements and forces at the two ends of the unit cell are denoted to be u_n and u_{n+1}, and F_n and F_{n+1}, respectively, as shown in Fig.1. By analyzing the dynamic response of the cell, the relation between the forces and the displacements at the ends of the unit cell can be determined. For $M=0$, the relation is

$$\frac{1}{2}(F_{n+1}+F_n)=\frac{1}{2}[k+\frac{k'}{1-\omega_0^2/\omega^2}](u_{n+1}-u_n), \qquad F_{n+1}-F_n=-\frac{m\omega^2}{1-\omega^2/\omega_1^2}\frac{u_{n+1}+u_n}{2} \tag{1}$$

where ω_1 *and* ω_0 are the natural frequencies of the unit cell, given by $\omega_1^2=2(k'+k)/m$, $\omega_0^2=2R^2k'/(G^2m)$, with R being given in Fig.1 and G being the radius of gyration of the body, i.e. $I=mG^2$. It can be observed from Eq. 1 that the effective modulus and mass of the unit cell are

$$k_e=\frac{1}{2}\left[k+\frac{k'}{1-\omega_0^2/\omega^2}\right] \qquad m_e=\frac{m}{1-\omega^2/\omega_1^2} \tag{2}$$

The effective modulus and mass of the unit cell could be negative under certain frequencies, because of the resonance behaviour of the unit cell. Negative modulus is obtained when

$$\omega_0^2>\omega^2>\frac{k}{k+k'}\omega_0^2 \tag{3}$$

while negative mass is achieved when

$$\omega^2>\omega_1^2=\frac{2(k+k')}{m} \tag{4}$$

Double negative, i.e. negative modulus and negative mass, can be obtained when $\omega_0>\omega>\omega_1$ if $\omega_0>\omega_1$, which is equivalent to $\frac{(k+k')G^2}{R^2k'}<1$. If the central spring is removed, i.e. $k=0$, the double negative condition is then simply $G/R<1$, with G/R being the ratio between the radius of gyration and the distance between the mass centre and the spring.

Dispersive relation and wave propagation

Consider now the propagation of a harmonic wave in the elastic metamaterial formed by repeated unit cells of length L. By using Eq. 1 and considering the consistency condition of the periodic cells, the governing equation for the wave propagation can be determined to be

$$k_e(u_{n+2}-2u_{n+1}+u_n)+\frac{m_e\omega^2}{4}(u_{n+2}+2u_{n+1}+u_n)=0 \tag{5}$$

For a harmonic wave propagating along the structure $u(x)=u_0e^{iKx}$ with K being the wave number, the following dispersive relation can be obtained from Eq. 5, which relates the wave number and the frequency of the wave,

$$\frac{(e^{iKL}-1)^2}{(e^{iKL}+1)^2}+\frac{m_e\omega^2}{4k_e}=0 \tag{6}$$

Case (i) $m_e/k_e>0$, when both m_e *and* k_e are positive or negative, the dispersion curve can be reduced to

$$KL=\pm\cos^{-1}(\frac{1-\gamma^2}{1+\gamma^2}) \tag{7}$$

with $\gamma=\omega\sqrt{|m_e/(4k_e)|}$. The wave number will be real, indicating that wave propagation will occur. It can be proved that the negative sign corresponds to the case where double negative is achieved, in which the direction of wave propagation is opposite to that of energy flow.

Case (ii) $m_e/k_e<0$, single negative when only one of m_e and k_e is negative,

$$iKL=\ln(\frac{1\pm\gamma}{1\mp\gamma}) \tag{8}$$

In this case the wave number is an imaginary number, indicating that there is no wave propagation.

RESULTS AND DISCUSSION

The existence of double negative response can be easily illustrated using the dispersion curve. For the special case where the central spring is removed, i.e. $k=0$, the dispersion curve is shown in Fig.2, in which the positive value of the wave number represents the imaginary part and the negative value is the real part. The existence of negative real solutions clearly shows the double negative response of the model between the two fundamental frequencies. For high frequencies, the negative effective mass results in the decay of the motion with position, while for low frequencies, the response shows strain-softening.

References

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