ON TRANSFORMATION METHODS FOR THE DESIGN OF METAMATERIALS

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Summary Metamaterials and cloaking technology have drawn much attention in recent years. The most powerful tool to design metamaterial is called the transformation method, such as transformation optics, transformation acoustics and transformation elastics. The theoretic basis of the transformation method is the form invariance property of the wave equation. Although such property has been clearly proved for transformation optics, the similar conclusion can only be obtained analogically for transformation acoustics, and it is still not very clear for transformation elastics so far. In this talk, I am going to discuss the transformation methods by using tensor analysis. It is concluded that if we rewrite the wave equations in a more general form, it is possible to prove the form invariance.

INTRODUCTION

If the wave trajectory could be controlled at will, many interesting devices, such as the perfect lens and cloaks, would be designed. Design of such devices is an inverse problem, which is usually characterized with non-unique solutions. However, since the two papers published side by side in Science magazine [1, 2], people find the transformation optics is a powerful tool to construct a possible solution for electro-magnetic devices. But it should be noted that the theoretic basis of transformation optics is the form invariance of Maxwell equations after arbitrary coordinate transformation.
Inspired by transformation optics, people successfully applied the transformation method to design acoustic devices [3, 4] and coined the name of transformation acoustics. Instead of giving a rigorous proof of the form invariance of the acoustic equation, people just analogically linked it with Maxwell equations.
However, people found it is more difficult to use the transformation method to design elastic devices, because the Navier equation does not observe the form invariance under arbitrary coordinate transformation [5]. Therefore, there still lacks a very good demonstration of general elastic cloaking, and only a few special cases were studied [6-10] or approximately achieved [11].
However, based on the definition of tensor, the wave equation in tensor form must be form invariant under arbitrary coordinate transformation. With this question in mind, I just want to discuss the form invariance of electromagnetic, acoustic and elastic wave equations using the tensor analysis tool. It is found that the form invariance does exist if we rewrite wave equations in more general forms.

ELECTROMAGNETIC EQUATIONS

Without lose of generality, let us consider the equation:

$$\nabla \times \boldsymbol{E} + i\omega\mu\boldsymbol{H} = \mathbf{0} \tag{1.1}$$

where $\boldsymbol{E}$ is the vector of electric field intensity, $\boldsymbol{H}$ is the vector of magnetic field intensity and ω is the wave frequency. The general index form with covariant basis of Eq. (1.1) is:

$$\varepsilon^{ijk} E_{j;i} + i\omega\mu g^{kl} H_l = 0 \tag{1.2}$$

where ε^{ijk} is contravariant component of Eddington tensor; $_{;i}$ is covariant derivative and g^{kl} is the contravariant component of metric tensor. Under an arbitrary coordinate transformation from x^i to $x^{i'}\left(x^i\right)$, we have the relations:

$$g^{kl} = \beta^k_{k'}\beta^l_{l'} g^{k'l'}, \qquad E_{j;i} = \beta^{j'}_j \beta^{i'}_i E_{j';i'}, \qquad H_l = \beta^{l'}_l H_{l'} \tag{1.3}$$

where $\beta^{i'}_i = \partial x^{i'} / \partial x^i$ is the component of deformation gradient $\boldsymbol{\beta}$. (Other possibilities could exist for Eq. (1.3) [12])
Substituting Eq. (1.3) into Eq. (1.2), yields:

$$\varepsilon^{i'j'k'} E_{j';i'} + i\omega\mu g^{k'l'} H_{l'} = 0 \tag{1.4}$$

Comparing Eq. (1.2) with Eq. (1.4), it is clear that the form of Maxwell's equations is invariant after arbitrary coordinate transformation.
Notice that $\varepsilon^{i'j'k'} = \sqrt{\det\left(g^{i'j'}\right)} e^{i'j'k'}$, where $e^{i'j'k'}$ is the Levi-Civita symbol. Thus, Eqs. (1.2) and (1.4) can be rewritten as:

$$e^{ijk} E_{j;i} + i\omega\left(\mu g^{kl} / \sqrt{\det\left(g^{ij}\right)}\right) H_l = 0 \tag{1.5}$$

$$e^{i'j'k'} E_{j';i'} + i\omega\left(\mu g^{k'l'} / \sqrt{\det\left(g^{i'j'}\right)}\right) H_{l'} = 0 \tag{1.6}$$

Since Eq. (1.5) and Eq. (1.6) are in the same form, one can obtain the effective magnetic permeability of the original configuration and the transformed configuration as: $\mu^{kl} = \mu g^{kl} / \sqrt{\det\left(g^{ij}\right)}$ and $\mu^{k'l'} = \mu g^{k'l'} / \sqrt{\det\left(g^{i'j'}\right)}$.

If all material points are described in global Cartesian coordinate system, then Eqs. (1.5) and (1.6) becomes $e^{ijk}E_{j,i}+i\omega\mu\delta^{kl}H_l=0$ and $e^{i'j'k'}E_{j',i'}+i\omega\left(\mu g^{k'l'}/\sqrt{\det\left(g^{i'j'}\right)}\right)H_{l'}=0$, respectively.

ACOUSTIC EQUATION

Acoustic wave follows a simple scalar equation:

$$\nabla\cdot\left(\nabla p\right)+\frac{\rho}{K}\omega^2 p=0 \tag{2.1}$$

where p is the pressure, K is the bulk stiffness and ρ is the mass density. Similar to above deduction, we can write the acoustic equation in general index form with contravariant components (covariant basis) as:

$$g^{ij}\left(p_{;i}\right)_{;j}+\frac{\rho}{K}\omega^2 p=0\text{, or }\left(p_{;i}\right)_{;j}+\frac{\rho g_{ij}}{K}\omega^2 p=0 \tag{2.2}$$

After arbitrary coordinate transformation, it becomes:

$$g^{i'j'}\left(p'_{;i'}\right)_{;j'}+\frac{\rho}{K}\omega^2 p'=0\text{, or }\left(p'_{;i'}\right)_{;j'}+\frac{\rho g_{i'j'}}{K}\omega^2 p'=0 \tag{2,3}$$

Comparing Eq. (2.2) with Eq. (2.3), it is clear that the form of acoustic equation is invariant after arbitrary coordinate transformation. And generally, the density is a second order tensor.

ELASTIC EQUATION

The Navier equation for elastic wave is:

$$\nabla\cdot\left(\boldsymbol{C}:\nabla\boldsymbol{u}\right)+\rho\omega^2\boldsymbol{u}=\boldsymbol{0} \tag{3.1}$$

where $\boldsymbol{C}$ is the elastic tensor and $\boldsymbol{u}$ is the displacement vector. Similar to above deduction, we can write the elastic equation in general index form with contravariant components (covariant basis) as:

$$\left(C^{ijkl}u_{i;j}\right)_{;k}+\rho\omega^2 g^{lm}u_m=0 \tag{3.2}$$

After arbitrary coordinate transformation, it becomes:

$$\left(\frac{1}{\sqrt{\det\left(g^{i'j'}\right)}}\beta_i^{i'}\beta_j^{j'}\beta_k^{k'}\beta_l^{l'}C^{ijkl}u_{i';j'}\right)_{;k'}+\frac{1}{\sqrt{\det\left(g^{i'j'}\right)}}\rho\omega^2 g^{l'm'}u_{m'}=0 \tag{3.3}$$

Comparing Eq. (3.2) with Eq. (3.3), it is clear that the form of elastic equation is invariant after arbitrary coordinate transformation. And generally, the density is a second order tensor.

CONCLUSIONS

As the above discussion, if we rewrite the wave equations in a more general form, it is possible to prove the form invariance. Furthermore, if conformal mapping is used, the resultant metamaterial is isotropic [10, 13], which is more desirable. More details and numerical examples will be given in the oral presentation.

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