THE CONSISTENCY BETWEEN THE GAS KINETIC UNIFIED ALGORITHM AND THE DSMC METHOD IN DESCRIBING THE BOLTZMANN EQUATION

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Summary This study investigates the consistency between the gas kinetic unified algorithm and the DSMC method in describing the Boltzmann equation. It starts from the equation to which the particle velocity distribution function in DSMC converges, by means of mathematical analysis, and arrives at the starting points of the gas kinetic unified algorithm. Numerical validations are presented, including the Couette flow, normal shock wave, 2D flow around a plate and 3D flow around a ball, and they support our conclusions.

Introduction

The basic equation in the gas kinetic theory, the Boltzmann equation, is an integral-differential equation. The highly nonlinear and multi-dimensional collision term brings formidable obstacles to its precise solution, and its numerical solution has always been a hot topic [1].

The DSMC method [2] takes advantage of the collision-relaxation feature of the Boltzmann equation, decouples the molecular motions and their collisions. Its represents the massive real molecules by a set of finite simulating particles and investigate their motions, collisions and interactions with the boundaries. Macroscopic variables are obtained by averaging the corresponding microscopic information. It has been proved [3] that the particle velocity distribution function of the DSMC method converges to a modified form of the Boltzmann equation when the number of simulating particles tends to infinites.

The gas kinetic unified algorithm [4,5] makes use of the Shakhov model [6] and develops the gas-kinetic velocity ordinate method. It converts the equation of velocity distribution function to hyperbolic conservation forms with nonlinear source terms, and adopts Hua-Wang integration method of number theory to deal with high-dimensional integration.

Since they both describe the Boltzmann equation, this study investigates their consistency. It starts from the equation to which the particle velocity distribution function in DSMC converges, by means of mathematical analysis, and arrives at the starting points of the gas kinetic unified algorithm. Numerical validations are presented, including Couette flow, normal shock wave, two dimensional flow around a plate and three dimensional flow around a ball. These examples support our conclusions. It should be noted that, the gas kinetic unified algorithm can be applied to a broader range.

Theoretical derivations

When the number of the simulating particles turns to infinite, the velocity distribution function of the DSMC converges to

$$\frac{\partial}{\partial t} f(t,x,v) + (v,\nabla_x) f(t,x,v) = J_{mol} f(t,x,v) \tag{1}$$

with

$$J_{mol} f(t,x,v) = \int_G dy \int_{R^3} dw \int_{S^2} de\, h(x,y) q(v,w,e) \left[f(t,x,v^*) f(t,y,w^*) - f(t,x,v) f(t,y,w) \right]$$

The collision term can be decomposed into the depleting term and the replenishing term

$$J_{repl} = \int_G dy \int_{R^3} dw \int_{S^2} de\, h(x,y) q(v,w,e) f(t,x,v^*) f(t,y,w^*), \tag{2}$$

$$J_{depl} = \int_G dy \int_{R^3} dw \int_{S^2} de\, h(x,y) q(v,w,e) f(t,x,v) f(t,y,w). \tag{3}$$

If the collision frequency is defined as

$$\nu = \int_G dy \int_{R^3} dw \int_{S^2} de\, h(x,y) q(v,w,e) f(t,y,w), \tag{4}$$

the depleting term reduces to

$$J_{depl} = \nu f(t,x,v). \tag{5}$$

Meanwhile, if the cell size turns to infinitesimal, the replenishing term can be written as

$$J_{repl} = \int_{R^3} dw \int_{S^2} deq(v,w,e) f(t,x,v^*) f(t,x,w^*) + O(\varepsilon) \tag{6}$$

for arbitrary small amount of ε.

Assume

$$f^+(t,x,v) = \frac{1}{\nu} \int_{R^3} dw \int_{S^2} deq(v,w,e) f(t,x,v^*) f(t,x,w^*), \tag{7}$$

equation (1) turns to

$$\frac{\partial}{\partial t} f(t,x,v) + (v,\nabla_x) f(t,x,v) = \nu \left[f^+(t,x,v) - f(t,x,v) \right] + O(\varepsilon). \tag{8}$$

Neglecting the high order term, this is where the gas kinetic unified algorithm starts from.

For the collision frequency, the Maxwell model gives

$$\nu = \frac{2W_{0,m}^2}{3A_2(5)} \frac{nkT}{\mu}. \tag{9}$$

Appropriate choice of $W_{0,m}$ yields to traditional expression of the collision frequency. Meanwhile, Chapmann-Enskog expansion leads to

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$$f^{+}(t,x,v)=f^{(0)}\left[1+(1-\Pr)\vec{c}\bullet\vec{q}\left(\frac{c^2}{RT}-5\right)/(5PRT)\right] \tag{10}$$

with $f^{(0)}$ representing local equilibrium distribution function.

Numerical validations

Numerical tests are carried out using Couette flow, normal shock wave, two dimensional flow around a plate and three dimensional flow around a ball. Typical illustrations are given in the following figures. These results support the consistency between these two approaches. It should be noted that the unified algorithm removes the cell size limitation and can deal with flows from rarefied to continuum, while the DSMC is mainly for rarefied gas dynamics.

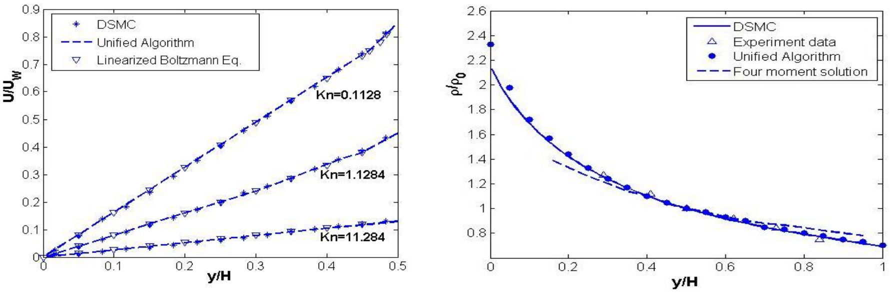

Figure 1. Velocity(left) and density(right) distribution in the Couette flow, including results of DSMC, the gas kinetic unified algorithm, the four moment method as well as experiment data.

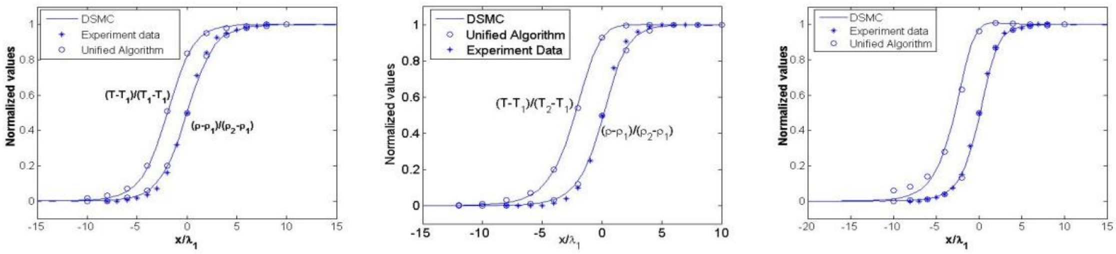

Figure 2. Density and temperature distribution with shock Mach numbers 2.0(left), 3.8(middle) and 9.0(right), including results of DSMC and the gas kinetic unified algorithm as well as experiment data.

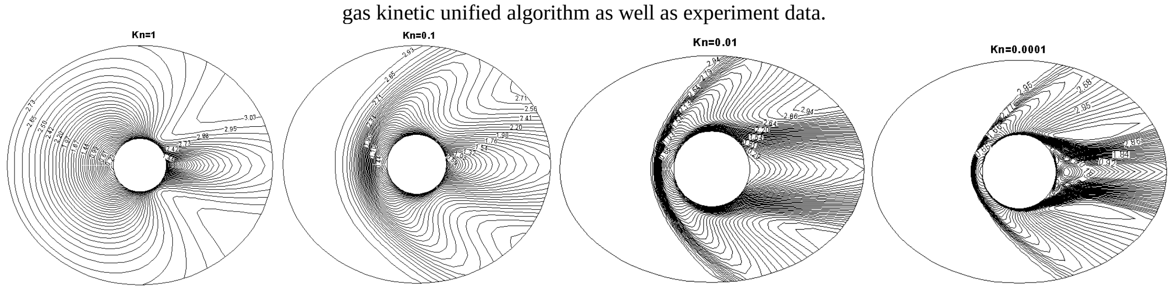

Figure 3. Mach number contours obtained by the gas kinetic unified algorithm for the flow around a ball, from rarefied to continuum flow.

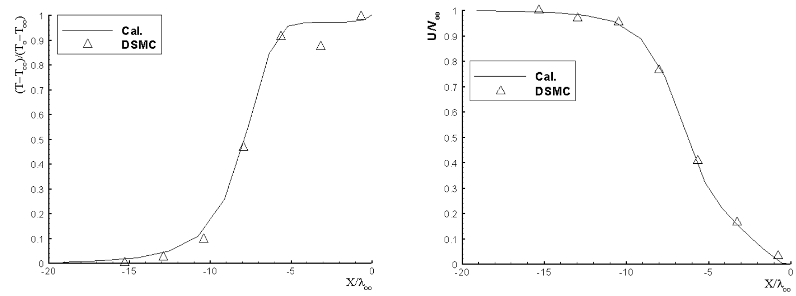

Figure 4. Temperature (left) and density(right) distribution along the stagnation line for flow around a ball（$Kn_\infty = 0.03$、$Ma_\infty = 3.83$）.

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