

A GENERALIZED RECIPROCAL THEOREM FOR PREDICTING THE FORCE AND TORQUE ON BODIES MOVING IN AN INHOMOGENEOUS FLOW AT ARBITRARY REYNOLDS NUMBER

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Summary We present the main ideas behind the derivation and some applications of a theorem that parallels Lorentz's reciprocal theorem and provides general expressions for the force and torque acting on a rigid body of arbitrary shape moving in an inhomogeneous incompressible flow at arbitrary Reynolds number. We show that this theorem allows a clear physical interpretation of the various contributions to the loads and encompasses all results available in both limits of inviscid and creeping flows.

Predicting the forces and torques acting on bodies moving in arbitrary flow fields has always been a central concern in Fluid Mechanics. Specific formulations of the problem have been developed in the two limits where the governing equations become linear, namely Stokes flows and potential flows. However the general case where inertial and viscous effects are both present poses much greater difficulties, owing to the nonlinear interplay between the various contributions. In general, no closed-form expression of the loads can be expected in this regime and the best that can be achieved is to express them as the sum of various surface and volume integrals involving solely the velocity field $\mathbf{U}$ and its derivatives, especially the local vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{U}$. The situation may however be more favourable with high-Reynolds-number bubbles because the boundary layer induced by the shear-free condition at a bubble surface only induces secondary changes in the flow field.

The work described in this communication summarizes the main findings of a recent investigation [1] in which the above goal has been pursued with the objective of obtaining general expressions for the loads on a body of arbitrary shape that obeys either a no-slip or a shear-free condition at its surface S_B and moves in an inhomogeneous flow field with a translational velocity $\mathbf{V}(t)$ and a rotation rate $\boldsymbol{\Omega}(t)$, flow inhomogeneity being either due to velocity gradients in the carrying flow $\mathbf{U}_U(\mathbf{x}, t)$ or to the presence of a bounding wall S_W (figure 1). The approach we follow stems directly from the complete Navier-Stokes equations and parallels the well-known Lorentz reciprocal theorem widely used in low-Reynolds-number hydrodynamics. It appears to be the proper way to derive rationally the generalized Kelvin-Kirchhoff equations to predict all components of the force and torque acting on a rigid body moving in an incompressible inhomogeneous flow at arbitrary Reynolds number. The key to this approach, which extends that of [2] and [3], stands in the use of auxiliary velocity fields $\hat{\mathbf{U}}$ which are both incompressible and irrotational and correspond to the flow disturbance induced by a unit translational velocity $\hat{\mathbf{i}}$ or rotation rate $\hat{\mathbf{j}}$ of the body moving in a fluid at rest. Such auxiliary fields are exact solutions of the full Navier-Stokes equations since the viscous stress associated with an irrotational velocity field $\hat{\mathbf{U}} = \nabla \hat{\Phi}$ has zero flux. By properly selecting the orientation of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, all required components of the loads can be evaluated, irrespective of their orientation with respect to the relative motion between the body and fluid.

When the fluid is at rest at infinity and the vorticity distribution is compact within the flow domain V , suitable combinations of the governing equations and proper use of Green and Stokes theorems provide each component of the hydrodynamic force $\mathbf{F}$ in the form

$$\begin{aligned} \hat{\mathbf{i}} \cdot \mathbf{F} = & \frac{d}{dt} \int_{S_B} \hat{\Phi} \mathbf{W} \cdot \mathbf{n} dS + \int_V \{(\boldsymbol{\omega} + \boldsymbol{\omega}_B) \times \mathbf{U}\} \cdot (\hat{\mathbf{W}} - \hat{\mathbf{U}}) dV \\ & - \frac{1}{Re} \int_{S_B} [(\hat{\mathbf{U}} - \hat{\mathbf{W}}) \times \boldsymbol{\omega}] \cdot \mathbf{n} - \int_{S_W} \left\{ \frac{1}{2} (\mathbf{U} \cdot \mathbf{U}) \hat{\mathbf{W}} \cdot \mathbf{n} + \frac{1}{Re} (\hat{\mathbf{U}} \times \boldsymbol{\omega}) \cdot \mathbf{n} \right\} dS, \end{aligned} \quad (1)$$

where $\mathbf{W} = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}$ and $\hat{\mathbf{W}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} \times \mathbf{r}$, $\mathbf{r}$ being the local position from the body centroid and $\mathbf{n}$ the local unit normal directed towards the fluid interior. Expression (1) and its counterpart for the torque achieve a clear splitting between the various contributions to the loads: in (1), the two integrals on S_B correspond to the added-mass and viscous force, respectively, while the volume integral is the vortex-induced contribution which involves the vorticity $\boldsymbol{\omega}$ in the bulk but also the bound vorticity $\boldsymbol{\omega}_B$ corresponding to the difference on S_B between the tangential component $\mathbf{n} \times \mathbf{U}$ of the actual velocity field and that of the virtual irrotational velocity field $\nabla \hat{\Phi}$ that satisfies the same kinematic condition on S_B and S_W . The wall provides an inertial contribution normal to itself and an additional viscous force. When the Reynolds number goes to zero, only contributions proportional to Re^{-1} remain in (1), possibly supplemented by the added-mass term if unsteadiness is large.

As an application of this formulation, we shall discuss the case on a pair of spherical bubbles translating side-by-side at high Reynolds number. We will show that, by successively selecting $\hat{\mathbf{i}}$ to be along the path and along the line of centres, one recovers the leading $O(Re^{-1})$ expression for the viscous drag (including corrections due to the interaction with the second bubble) and the inertial transverse attractive force. The viscous correction due to the boundary layer will also be shown to be of $O(Re^{-3/2})$ in the direction of the motion and of $O(Re^{-1})$ in the transverse direction, a result which could not be obtained with available integral approaches but was recently inferred from direct numerical simulations.

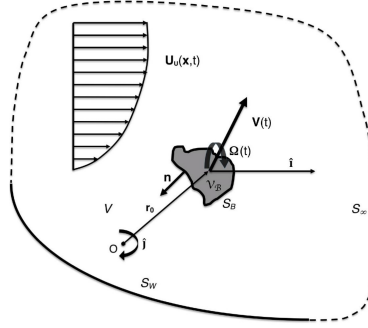


Figure 1. Sketch of the general flow configuration.

When the body moves in a nonuniform carrying flow, new contributions arise, in addition to the net force and torque exerted by the undisturbed flow on the volume of fluid $\mathcal{V}_B$ occupied by the body. To make all couplings between the undisturbed velocity gradients and the body translational and rotational velocities explicit, we consider the case of a general linear carrying flow $\mathbf{U}_U(\mathbf{x}, t) = \mathbf{U}_0 + \mathbf{r} \cdot \nabla \mathbf{U}_0$ in the weakly inhomogeneous limit where the undisturbed velocity varies over distances much larger than the body size. Setting $\tilde{\mathbf{U}} = \mathbf{U} - \mathbf{U}_0$, $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \boldsymbol{\omega}_0$, etc. and expanding the various contributions involving the undisturbed strain rate $\mathbf{S}_0$ and vorticity $\boldsymbol{\omega}_0$, we show for instance that the net force on a non-rotating body moving in an unbounded flow domain is the sum of the viscous contribution on S_B and of the vortical contribution in V as given in (1) plus an extra force $\mathbf{F}_0$ due to the underlying linear flow which writes

$$\begin{aligned} \hat{\mathbf{i}} \cdot \mathbf{F}_0 &= \hat{\mathbf{i}} \cdot \left\{ \frac{D_0 \mathbf{U}_0}{Dt} \mathcal{V}_B - \mathbf{A} \cdot \left(\frac{d\mathbf{V}}{dt} - \frac{D_0 \mathbf{U}_0}{Dt} \right) + [\mathbf{A} \cdot \mathbf{S}_0 - \mathbf{S}_0 \cdot \mathbf{A}] \cdot \tilde{\mathbf{V}} + \left(\frac{\mathbf{I}}{2} + \frac{\mathbf{A}}{\mathcal{V}_B} \right) \cdot [\boldsymbol{\omega}_0 \times (\mathbf{A} \cdot \tilde{\mathbf{V}})] - \frac{1}{2} \mathbf{A} \cdot (\tilde{\mathbf{V}} \times \boldsymbol{\omega}_0) \right\} \\ &- \int_V \{ \hat{\phi} \boldsymbol{\omega}_0 \cdot (\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}_B)_0 + 2(\tilde{\mathbf{U}}_0 - \nabla \tilde{\Phi}_0) \cdot \mathbf{S}_0 \cdot \hat{\mathbf{U}} \} dV, \end{aligned} \quad (2)$$

where D_0/Dt denotes the material derivative following $\mathbf{U}_0$, $\mathbf{I}$ is the unit tensor and $\mathbf{A}$ is the translational added-mass tensor. The terms in closed form in the r.h.s. of (2) may receive a clear physical interpretation. In particular the last two of them provide a contribution to the lift force, the first of them being nonzero only when the principal axes of the body are not aligned with those of $\mathbf{S}_0$. The volume integral in (2) results from the distortion of the vortical disturbance due to the body by the underlying strain rate and vorticity. It vanishes in the special case of inviscid two-dimensional flows and is negligibly small at short time in inviscid three-dimensional flows. The present formulation recovers all available inviscid predictions of [4] (which themselves encompass the original theoretical findings of [5]), and [6] for irrotational and vortical undisturbed flows, respectively. Also, all available inviscid predictions for spheres and circular cylinders immersed in linear flows, such as those of [7], appear as particular cases of (2). As an application of (2) we will discuss the inviscid lift force acting on circular cylinder translating steadily along a plane wall parallel to a linear shear flow, and will show that if the cylinder lags behind the fluid, there is a critical distance to the wall at which the net lift force vanishes.

To conclude we will emphasize that, beyond theoretical predictions with high-Reynolds-number bubbles or inviscid situations, the present theorem may be useful in the context of experimental studies at finite Reynolds number, especially those in which the determination of the pressure distribution at the body surface is difficult, such as with flapping wings or fins. Indeed, once the velocity and vorticity distributions have been determined (using Particle Image Velocimetry for instance), this theorem provides a direct tool for evaluating the various contributions to the loads. We will also indicate some directions to generalize this approach, in particular to deformable bodies and flows with density gradients.

References

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