

A NEW SOLUTION OF EULER'S EQUATION OF MOTION WITH HELICITY

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Summary On the basis of the principle of least action, a new representation of solution is derived for the velocity field of rotational flows of a compressible ideal fluid. The velocity field is represented in general by scalar potentials and vector potentials of frozen field, *i.e.* the latter potentials is convected with the fluid flow under effect of stretching. It is verified that the system of new expressions *in fact* satisfies the Euler's equation of motion. The Lagrangian for the action consists of main terms of total kinetic energy and internal energy (with negative sign), together with two terms yielding the equations of continuity and entropy and the third term which yields a *new rotational component* of velocity field. This solution gives an explicit expression of non-vanishing helicity, and improves the classical Clebsch-type solution in the sense that the Clebsch potentials are simply convected by the flow (without stretching effect).

INTRODUCTION

Fluid mechanics is a field theory of Newtonian mechanics of Galilean symmetry. Two symmetries are known as sub-groups of the Galilean group: translation (space and time) and space-rotation. A symmetry of a physical system means invariance with respect to a certain group of transformations and plays an essential role in the gauge theory of theoretical physics. Guided by the gauge theory, Kambe [1, 2] studied *rotational* flows of an ideal compressible fluid and investigated consequence of both global and local invariances of the fields in the space-time $(\mathbf{x}, t)$. Present study is focused on resolving a difficulty inherent in the traditional Lagrangian formulation.

It is as follows. Under the Eulerian variation in which variations are taken independently for all the field variables of *Eulerian description*, the principle of least action yields a general solution equivalent to the classical Clebsch solution (Clebsch [3], [1]). In this solution the vorticity has a special form such that the helicity vanishes. In a particular case of isentropic fluid in which the entropy s is uniform, the flow field thus obtained becomes *irrotational* (*see below*). This is a weak point of the traditional formulation, because even in such an isentropic fluid, the fluid flow should support rotational velocity fields. In addition, most traditional formulations of the action principle take into account both the continuity equation and isentropic condition as constraint conditions for variations. To do it, Lagrange multipliers are used. This is a mathematical artifact since physical meaning of the multipliers is not clear.

On the basis of the present gauge-theoretic formulation mentioned above, it is particularly remarkable that the convective derivative D_t defined by $D_t \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$ (the Lagrange derivative) is in fact the *covariant derivative* which is a building block of the gauge theory, where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field, $\nabla = (\partial_i)$ and $\partial_i \equiv \partial/\partial x^i$. In the present formulation, a new term is introduced in the Lagrangian for rotational motion of an ideal fluid.

VARIATIONAL FORMULATION

Total Lagrangian consists of main terms of total kinetic energy and internal energy ϵ (with negative sign), together with two terms yielding the equations of continuity and entropy and the third term which yields a new *rotational* component of velocity field.^{b)} Thus the total Lagrangian L , the action J and the Lagrangian density Λ are defined by

$$L = \int_V \Lambda(\mathbf{v}, \rho, s, \phi, \psi) d^3\mathbf{x}, \quad J = \int_{t_1}^{t_2} L dt = \int \int \Lambda dt d^3\mathbf{x}, \quad (1)$$

$$\Lambda = \frac{1}{2} \rho \langle \mathbf{v}, \mathbf{v} \rangle - \rho \epsilon(\rho, s) - \rho D_t \phi - \rho s D_t \psi - \langle \mathcal{L}_t^*[\mathbf{A}], \boldsymbol{\Omega} \rangle, \quad (\text{with the property, } \nabla \cdot \boldsymbol{\Omega} = 0, \nabla \cdot \mathbf{A} = 0), \quad (2)$$

where V is a domain in the $\mathbf{x}$ -space (chosen arbitrarily), $\langle \cdot, \cdot \rangle$ denotes the inner product, $\rho(\mathbf{x}, t)$ and $s(\mathbf{x}, t)$ are the fluid density and specific entropy (*per unit mass*), and $\phi(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$ are scalar potentials associated with mass and entropy respectively. The last term $\langle \mathcal{L}_t^*[\mathbf{A}], \boldsymbol{\Omega} \rangle$ is *new* [1]. Its form is determined so as to satisfy the symmetries, *i.e.* invariance with respect to translation (space and time) and space-rotation. Lie-derivatives of a tangent vector $\boldsymbol{\Omega} = (\Omega^i)$ and a cotangent vector $\mathbf{A} = (A_i)$ are defined in the footnote^{c)}. Substituting the varied variables $\mathbf{v} + \delta\mathbf{v}$, $\rho + \delta\rho$, $s + \delta s$, $\phi + \delta\phi$ and $\psi + \delta\psi$ into $\Lambda(\mathbf{v}, \rho, s, \phi, \psi)$ and writing its variation as $\delta\Lambda$, we obtain

$$\delta\Lambda = \Lambda_{\mathbf{v}} \cdot \delta\mathbf{v} + \Lambda_{\rho} \delta\rho + \Lambda_s \delta s + \Lambda_{\phi} \delta\phi + \Lambda_{\psi} \delta\psi + \Lambda_{\boldsymbol{\Omega}} \cdot \delta\boldsymbol{\Omega} + \Lambda_{\mathbf{A}} \cdot \delta\mathbf{A} + \partial_t(\Lambda_t) + \partial_i(\Lambda_i), \quad (3)$$

where $\mathbf{p} \cdot \mathbf{q}$ denotes the scalar product of vectors $\mathbf{p}$ and $\mathbf{q}$. The action principle is $\delta J = \int dt \int d^3\mathbf{x} \delta\Lambda = 0$. The variations $\delta\mathbf{v}$, $\delta\rho$, $\dots$ are assumed to vanish on the boundary surface enclosing the domain V and at the end points of t_1 and t_2 , which make the last two terms $\partial_t(\Lambda_t) + \partial_i(\Lambda_i)$ of $\delta\Lambda$ vanish. By substituting the above, we must have $\Lambda_{\mathbf{v}} = 0$, $\Lambda_{\rho} = 0$ and $\Lambda_s = 0$ from the action principle for independent variations $\delta\mathbf{v}$, $\delta\rho$ and δs . Thus we obtain

$$\mathbf{v} = \nabla \phi + s \nabla \psi + \mathbf{w}, \quad \mathbf{w} \equiv \frac{1}{\rho} \boldsymbol{\Omega} \times (\nabla \times \mathbf{A}), \quad (4)$$

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^{b)} It can be shown by using the results of variational principle that each of these three additional terms can be expressed in the form of Lie derivative with respect to time t .

^{c)} $(\mathcal{L}_t[\boldsymbol{\Omega}])^i \equiv \partial_t \Omega^i + v^k \partial_k \Omega^i - \Omega^k \partial_k v^i$, and $(\mathcal{L}_t^*[\mathbf{A}])_i \equiv \partial_t A_i + v^k \partial_k A_i + A_k \partial_i v^k$.

$$\frac{1}{2} v^2 - h - \mathbf{D}_t \phi - s \mathbf{D}_t \psi = 0, \quad \mathbf{D}_t \psi + T = 0, \quad (5)$$

where $h = \epsilon + p/\rho$ is the specific enthalpy, p the pressure, T the temperature, and standard relations of thermodynamics are used.^{d)} In regard to the terms $\Lambda_\phi \delta\phi + \Lambda_\psi \delta\psi$, we must have $\Lambda_\phi = 0$ and $\Lambda_\psi = 0$. These lead to

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \partial_t s + \mathbf{v} \cdot \nabla s = \mathbf{D}_t s = 0. \quad (6)$$

Thus, we obtain the equation of continuity and the entropy equation. From $\Lambda_\Omega = 0$ and $\Lambda_A = 0$, we obtain

$$\mathcal{L}_t^*[A] = 0, \quad \partial_t \Omega + \nabla \times (\Omega \times \mathbf{v}) = 0. \quad (7)$$

It is seen that the vector potential Ω satisfies the equation of frozen field, and the Lie derivative of A vanishes.

Thus, we have obtained the results (4), (5), (6) and (7) from the variational principle. In particular, the third term $\mathbf{w}$ of the velocity (4) is new. It can be shown that the set of equations (4), (5), (6) and (7) satisfy the Euler's equation of motion. In fact, applying the derivative $\mathbf{D}_t (= \partial_t + \mathbf{v} \cdot \nabla)$ to $\mathbf{v}$ of (4), we have

$$\mathbf{D}_t[\mathbf{v}] = \mathbf{D}_t \nabla \phi + \mathbf{D}_t(s \nabla \psi) + \frac{1}{\rho} \mathbf{D}_t \mathbf{w} - \frac{1}{\rho^2} (\mathbf{D}_t \rho) \mathbf{w} = \nabla (\mathbf{D}_t \phi - \frac{1}{2} v^2 - sT) + T \nabla s,$$

where the second of (5), (6) and (7) are used. Finally, this reduces to the Euler's equation of motion owing to (5): $\mathbf{D}_t \mathbf{v} = -\nabla h + T \nabla s = -(1/\rho) \nabla p$, since $dh = (1/\rho) dp + T ds$ by the thermodynamics. Thus it is found that the present *Eulerian variation* has lead to the *correct* result.

NEW ASPECTS OF THE PRESENT FORMULATION

In particular, the present solution is new in the following two aspects at least.

(i) A new aspect of the present expression becomes clear if we neglect the third term $\mathbf{w}$ of velocity (4) derived from the new term $\langle \mathcal{L}_t^*[A], \Omega \rangle$ of the Lagrangian density (2). In this case, the velocity is given by $\mathbf{v}_0 = \nabla \phi + s \nabla \psi$. Taking its curl, the vorticity is given by $\omega_0 = \nabla \times \mathbf{v}_0 = \nabla s \times \nabla \psi$. Then the helicity is given as follows (where $\nabla \cdot \omega_0 = 0$ is used),

$$H = \int \mathbf{v}_0 \cdot \omega_0 d^3x = \int (\nabla \phi + s \nabla \psi) \cdot (\nabla s \times \nabla \psi) d^3x = \int (\nabla \phi) \cdot \omega_0 d^3x = \int \nabla \cdot (\phi \omega_0) d^3x = 0,$$

if $\omega_0 = 0$ at large distances, or if $|\phi \omega_0| = O(|x|^{-3-\alpha})$ (with a positive parameter α) as $|x| \rightarrow \infty$. Furthermore, if the fluid is isentropic, *i.e.* $s = s_0$ (constant), we have $\omega_0 = 0$. Namely the flow is irrotational without the term $\mathbf{w}$.

However, in the present solution, we have $\text{curl} \mathbf{w} \neq 0$ in general. Therefore, it is *rotational* even in the isentropic fluid. In addition, the helicity does not vanish in general.

(ii) According to (7), the potentials A_i (a cotangent vector) and Ω^i (a tangent vector) of the present solution satisfy the following equations,

$$\partial_t A_i + (\mathbf{v} \cdot \nabla) A_i = -A_k \partial_i v^k, \quad \partial_t \Omega^i + (\mathbf{v} \cdot \nabla) \Omega^i = \Omega^k \partial_k v^i - \Omega^i \partial_k v^k. \quad (8)$$

These equations are essentially different in character from the equations of the Clebsch potentials. In fact, the Clebsch-type solution is expressed by the following,

$$\begin{aligned} \mathbf{v} &= \nabla \phi + s \nabla \psi + \sum_i B_i \nabla C_i, & \frac{1}{2} v^2 + h + \partial_t \phi + s \partial_t \psi + \sum_i B_i \partial_t C_i &= 0, \\ \partial_t B_i + (\mathbf{v} \cdot \nabla) B_i &= 0, & \partial_t C_i + (\mathbf{v} \cdot \nabla) C_i &= 0, \quad (i = 1, 2). \end{aligned} \quad (9)$$

and $\mathbf{D}_t s = 0$, $\mathbf{D}_t \psi = 0$ [2, 3]. It is not difficult to show that this Clebsch solution satisfies the Euler's equation of motion, too. Comparing the two equations (8) and (9) for potentials, it is obvious that the right hand sides are different. In particular, the right hand sides vanish in the two equations of (9). Namely, the potentials B_i and C_i of the Clebsch solution are simply convected by the flow without change, while the tangent vector Ω^i of the present solution are frozen to the flow and stretched by the fluid motion. The cotangent vector A_i is also frozen to the flow by the constraint of vanishing Lie derivative $\mathcal{L}_t^*[A] = 0$ by (7).

CONCLUSIONS

An improvement of variational formulation is proposed for rotational flows of an ideal compressible fluid by introducing a new gauge-invariant term in Λ . The system of new expressions derived from the principle of least action satisfies the Euler's equation of motion. Therefore we have obtained a new expression of solution to the Euler's equation of motion.

References

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- [3] Clebsch A.: Über die Integration der hydrodynamischen Gleichungen. *J. Reine Angew. Math.*, **56**: 1–10, 1859.

^{d)} $(\partial \epsilon / \partial \rho)_s = p / \rho^2$, $(\partial / \partial \rho)_s (\rho \epsilon) = \epsilon + \rho (\partial \epsilon / \partial \rho)_s = \epsilon + p / \rho = h$, and $(\partial \epsilon / \partial s)_\rho = T$. Then we have $d\epsilon = (p / \rho^2) d\rho + T ds$ and $dh = (1 / \rho) dp + T ds$.