EVOLUTION OF WEAK SHOCKS IN ONE DIMENSIONAL PLANAR AND NON-PLANAR GASDYNAMIC FLOWS

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Summary Asymptotic decay laws for planar and nonplanar shock waves and the first order induced discontinuities that catch up with the shock from behind are obtained using four different approximation methods. The singular surface theory is used to derive a pair of transport equations for the shock strength and the associated first order discontinuity. The asymptotic behavior of both the discontinuities is completely analyzed. It is shown that for a weak shock, the precursor disturbance evolves like an acceleration wave at the leading order. We show that the asymptotic decay laws for weak shocks and the accompanying first order discontinuity are exactly the ones obtained by using the theory of nonlinear geometrical optics, the theory of simple waves using Riemann invariants, and the theory of relatively undistorted waves. It follows that the relatively undistorted wave approximation is a consequence of the simple wave formalism using Riemann invariants.

INTRODUCTION

The study of the evolutionary behaviour of nonlinear waves in diverse branches of continuum mechanics has long been a subject of great interest from both mathematical and physical points of view. A rigorous mathematical approach to describe the kinematics of a shock has been proposed by Maslov [1] using the theory of generalized functions. General properties and the instantaneous growth and decay behaviour of shock waves in elastic materials have been studied using the singular surface theory in a several articles (see for instance, Chen [2]). An interesting study on the evolutionary behaviour of shocks in elastic nonconductors has been carried out by Fu and Scott [3] using singular surface theory together with the method of multiple scales, method of weakly nonlinear geometrical optics and the shock fitting method. Evolutionary behaviour of shocks in fluids, using singular surface theory, has been discussed by Anile [4] and Sharma et al [5].

In the present paper we are concerned with planar, axially and radially symmetric flows of a polytropic gas behind a shock front which is propagating into a uniform region at rest; here, we employ four different approximation methods, namely, the singular surface theory, the method of weakly nonlinear geometrical optics, the characteristic approach using simple waves as kinematic waves, and the method of relatively undistorted waves to derive asymptotic decay laws for shocks and the accompanying first order discontinuity. The essential approximation involved in the method of weakly nonlinear geometrical optics is that the wave is of small amplitude and is slowly modulated; this means that the wave profile changes shape over distances or times much larger than a typical distance or time over which the wave profile itself varies. This method, unlike the method of characteristics, is applicable for problems involving several independent variables; indeed, a formal and systematic derivation of nonplanar simple wave solutions is possible only in the high frequency or geometrical-acoustics limit. Modulated simple wave theories have been developed by several authors; the extensive review by Seymour and Mortell [7] is particularly illuminating.

The other approximation method, namely the simple wave formulation using Riemann invariants is useful in the construction of solutions and development of singularities. For hyperbolic systems involving source terms or non-planar geometry, the governing equations may not admit exact simple wave solutions, but under small wave amplitude assumption, they may admit asymptotic modulated simple wave solutions. Indeed, hyperbolic systems are mathematically tractable using simple waves and Riemann techniques of gasdynamics, see for instance, Zheng [6]. Lastly, we consider the method of relatively undistorted waves; this applies to solutions of hyperbolic partial differential equations, and is based on the same slow modulation approximation used in the method of weakly nonlinear geometrical optics. But, unlike the method of weakly nonlinear geometrical optics, this theory makes no assumption on the magnitude of a disturbance (see, Varley and Cumberbatch [8] and Seymour and Mortell [7]).

MODEL AND THE BASIC EQUATIONS

We consider the one dimensional unsteady flow of a polytropic gas behind a planar, cylindrical or a spherical wave propagating into a uniform state at rest.The equations governing the flow are [9]

$$\rho_t + u\rho_x + \rho u_x + \tfrac{j}{x}\rho u = 0; \quad u_t + uu_x + \tfrac{1}{\rho}p_x = 0; \quad p_t + up_x + \rho a^2\left(u_x + \tfrac{ju}{x}\right) = 0, \tag{1}$$

where ρ is the density, u the gas velocity, p the pressure, and $a = (\gamma p/\rho)^{1/2}$ the sound speed with γ as the specific heat ratio of the gas; here t stands for time and x the distance being either axial in flows with planar $(j = 0)$ geometry, or radial in cylindrically symmetric $(j = 1)$ and spherically symmetric $(j = 2)$ flow configurations.

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As system (1) is quasilinear with each of these equations being a direct consequence of the corresponding conservation law, one expects shocks to appear in the flow after a finite running length or time. We consider the gas motion containing a shock wave propagating into a homogeneous quiescent equilibrium gas with intrinsic velocity U given by $U^{-1} = s'(x) = dt/\,dx$ where $t = s(x)$ denotes the location of the shock at position x.

MAIN RESULTS

Except some impirical methods, which have been developed in the past, no analytical method exists for describing the evolutionary behaviour of a shock wave without limiting its strength. In the present work, we generate a completely intrinsic description of plane, cylindrical and spherical shock waves of weak or even moderate strength, propagating into a uniform flow of a polytropic gas at rest. We use four different approximation methods to study the asymptotic evolutionary behaviour of shocks and the associated first order jump discontinuity, which may be interpreted as the disturbances that catch up with the shock from behind. Singular surface theory yields a set of coupled equations for the shock strength and the higher order jump discontinuities which couple the shock motion with the rearward flow. It is noticed that the transport equations for the shock strength $[p]$ and the first order jump discontinuity $[p_x]$ are nonlinear in $[p]$ and $[p_x]$ respectively; however, the transport equations for $[p_{xx}]$ is linear in $[p_{xx}]$, and this is true for all other higher order jump discontinuities. The transport equation for the shock strength shows that the evolutionary behaviour of the shock at any instant is influenced by the shock curvature and the first order jump discontinuity, representing the effect of precursor disturbances that overtake the shock from behind. This transport equation, in some sense, generalizes the CCW approximation [9], which although does not account for the effects of the disturbances that overtake the shock from behind, yet its success is unexpectedly remarkable for a particular class of problems. We show that the transport equation for the shock strength, in the limit of a weak shock and vanishing $[p_x]$, reduces exactly to the one obtained by the CCW approximation. Indeed, a pair of transport equations for $[p]$ and $[p_x]$, subjected to a truncation approximation by neglecting the second order jump discontinuity $[p_{xx}]$ yields a closed system consisting of only a pair of coupled ODEs, which can be regarded as a good approximation for the hierarchy of the system governing shock evolution. These equations are then solved for uniformly valid solutions using a singular perturbation technique that eliminates secular terms in the perturbed solution; the shock strength is assumed small, but the accompanying first order jump discontinuity may be finite (i.e., $\mathrm{O}(1)$) or small. It is shown that the precursor disturbances evolve to the leading order like an acceleration wave. Indeed, the decay of the induced discontinuity is much faster than the decay of the shock; it is noticed that for the case $[p_x] = \mathrm{O}(\epsilon)$, the shock decays faster as compared to the case when $[p_x] = \mathrm{O}(1)$. Asymptotic results for the decay of weak shocks fully agree with those given in [9].The second approximation method is that of weakly nonlinear geometrical optics; it is shown that this approximation method yields the same results as those from the previous method in their common range of validity. The next approximation method uses the simple wave formalism and the Riemann invariants [10]; it is shown that under the assumption of weak waves, this method also leads to the same results obtained using the first two approximation methods, namely

$$[p] \sim h\left(\frac{2}{(\gamma+1)k}\right)^{\frac{1}{2}} \begin{cases} x^{-1/2}, & \text{plane },\\ \frac{1}{\sqrt{2}}x^{-3/4}, & \text{cylindrical},\\ x^{-1}(\log x)^{-1/2}, & \text{spherical}; \end{cases} \tag{2}$$

and

$$[p_x] \sim h\left(\frac{2}{\gamma+1}\right) \begin{cases} x^{-1}, & \text{plane },\\ \frac{1}{2}x^{-1}, & \text{cylindrical},\\ (x\log x)^{-1}, & \text{spherical}, \end{cases} \tag{3}$$

as $x \to \infty$.

Lastly, we use the method of relatively undistorted waves. This approximation method makes no assumption on the magnitude of a disturbance and to the first order leads to the same evolution laws for the shock and the associated first order discontinuity; indeed, it transpires that the relatively undistorted wave approximation is a consequence of the simple wave formalism using Riemann invariants. Also a comparison with the CCW approximation is presented at the end.

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