INTERFACE STABILITY ANALYSIS OF MAGNETIC FLUID BY USING A METHOD FOR GENERAL USE AND NONLINEAR RESPONSE

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Summary The interface phenomena of magnetic fluid require rigorous as well as efficient analysis under arbitrary interface profiles and applied magnetic field distributions. Preceded by the magnetic analysis for this purpose, this paper presents a method to investigate various interface dynamic phenomena in the wavenumber space, and its application to a stability analysis. Branch lines at large interface elevation amplitude or its dependence on the relative permeability which previous works have not explored are shown.

EQUATION FOR INTERFACE MOTION

When we analyze the free surface phenomena of irrotational and inviscid magnetic fluid with all nonlinear effects but with no limitations on the interface profile, we use the following **equation for interface motion** which is derived from Bernoulli's equation and the dynamic boundary condition on the interface [1]:

$$\rho\frac{\partial\varphi}{\partial t}+S+p_0=0,\quad S\equiv D+G+C+T, \tag{1}$$

where ρ, φ, D, G, C, T, p_0 are the fluid density, velocity potential, dynamic pressure, gravity potential, surface tension, magnetic stress difference and atmospheric pressure, respectively. The velocity potential in (1) is obtained from the vertical component of the fluid velocity v_z as $\varphi{=}\int_{-\infty}^{\zeta}\mathrm{d}z v_z$, where ζ is the interface elevation. The magnetic stress difference $T{=}-\,[1/\mu_j]\{\mu_1\mu_2(h_X^2+h_Y^2)+b_Z^2\}/\,2$ represents the action from the magnetic field to the fluid, where μ_j denotes the permeability of the fluid (j=1) or the vacuum (j=2), $[\cdots]$ the difference of the value across the interface ($2{-}1$), respectively. The tangential magnetic field $h_{X,Y}$ and the normal magnetic flux b_Z should be obtained rigorously as well as efficiently under arbitrary interface profiles and applied magnetic field distributions [2].

The interface elevation $\zeta(\boldsymbol{R})$ and the sum of interface stresses $S(\boldsymbol{R})$, as well as other interface quantities, are regarded as the functions of the interface coordinate parameter $\boldsymbol{R}{=}(X,Y)$, and they are expanded into series of periodic functions. We rewrite (1) by the column vectors $\tilde{\boldsymbol{\zeta}}\equiv(\tilde{\zeta}_\mu)$ and $\tilde{\boldsymbol{S}}\equiv(\tilde{S}_\mu)$ composed of the expansion coefficients $\tilde{\zeta}_\mu$, $\tilde{S}_\mu$ belonging to the wavenumber vector $\boldsymbol{k}_\mu$. If the velocity field is harmonic and the component of v_z with the size of the wavenumber k is $(\partial\zeta/\partial t)\,\mathrm{e}^{kz}$, then the equation for $\tilde{\boldsymbol{\zeta}}$ is derived:

$$\frac{\partial^2\tilde{\boldsymbol{\zeta}}}{\partial t^2}=-\frac{k}{\rho}\tilde{\boldsymbol{S}}(\tilde{\boldsymbol{\zeta}}),\quad \tilde{\boldsymbol{S}}(\tilde{\boldsymbol{\zeta}})=\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{\zeta}})+\tilde{\boldsymbol{G}}(\tilde{\boldsymbol{\zeta}})+\tilde{\boldsymbol{C}}(\tilde{\boldsymbol{\zeta}})+\tilde{\boldsymbol{T}}(\tilde{\boldsymbol{h}}_X(\tilde{\boldsymbol{\zeta}}),\tilde{\boldsymbol{h}}_Y(\tilde{\boldsymbol{\zeta}}),\tilde{\boldsymbol{b}}_Z(\tilde{\boldsymbol{\zeta}})), \tag{2}$$

where $\tilde{\boldsymbol{D}}$, $\tilde{\boldsymbol{G}}$, $\tilde{\boldsymbol{C}}$ and $\tilde{\boldsymbol{T}}$ are column vectors composed of the expansion coefficients for D, G, C and T, respectively. Once the complex and nonlinear functional relation $\tilde{\boldsymbol{S}}(\tilde{\boldsymbol{\zeta}})$ is established, (2) is used not only for the stability analysis as here but also for investigating various interface dynamic phenomena in the wavenumber space.

STABILITY ANALYSIS

When an interface elevation ζ^0 is stable even if it is finite, the infinitesimal shift ζ^1 from ζ^0 approaches zero after a long time. From the difference of (2)'s for $\tilde{\boldsymbol{\zeta}}^0+\tilde{\boldsymbol{\zeta}}^1$ and $\tilde{\boldsymbol{\zeta}}^0$ corresponding to $\zeta^0{+}\zeta^1$ and ζ^0,

$$\frac{\partial^2\tilde{\boldsymbol{\zeta}}^1}{\partial t^2}\simeq-\frac{k}{\rho}\mathsf{H}(\tilde{\boldsymbol{\zeta}}^0)\tilde{\boldsymbol{\zeta}}^1,\quad \mathsf{H}(\tilde{\boldsymbol{\zeta}}^0)\equiv\begin{array}{c}{\scriptstyle 1\,\cdots\,\mu\,\cdots\,N}\\ \left(\left.\dfrac{\partial\tilde{\boldsymbol{S}}}{\partial\tilde{\zeta}_\mu}\right|_{\tilde{\boldsymbol{\zeta}}^0}\right)\end{array} \tag{3}$$

is derived, where the **gradient matrix** $\mathsf{H}(\tilde{\boldsymbol{\zeta}}^0)$ is a key quantity. We call here the minimum eigenvalue of $(k/\rho)\,\mathsf{H}(\tilde{\boldsymbol{\zeta}}^0)$ denoted by $h(k)$ as **nonlinear response**. The behavior of ζ^1 is investigated simply by (3).

The nonlinear response $h(k)$ is extended from the dispersion relation for linear wave as shown in Fig.2(a) [3], to be used for the cases of finite interface elevation and multimode. By changing the size of the wavenumber vector k of the mode providing the interface profile, we obtain $h(k)$ from the gradient matrix, and know the interface as unstable if the domain of $h(k){<}0$ appears in the k-space. This functional relation depends on the amplitude of the interface elevation ζ_0 as well as the applied magnetic field intensity H_0. We determine the critical magnetic field intensity $H_0{=}H_\mathrm{c}$ when the flat interface just changes unstable from the condition that the minimum of $h(k)$ is zero, that is, h=0 and $\partial h/\partial k$=0. The position of the minimum is the critical wavenumber k_c. We can draw a **branch line** on the (H_0,ζ_0) plane by connecting the points determined from H_c at each ζ_0.

INTERFACE PROFILE OF HEXAGONAL LATTICE

We investigate hereafter the stability of the interface profile with the hexagonal lattice as shown in Fig.1(a), for the magnetic fluid with the density $\rho{=}1.0\times10^3\,\mathrm{kg\,m^{-3}}$, capillary coefficient $\gamma{=}2.6\times10^{-2}\,\mathrm{N\,m^{-1}}$, and relative permeability

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μ_1/μ_0=1.1, 1.2, 1.3, 1.4 (μ_2/μ_0=1.0, μ_0=$4\pi\times10^{-7}$ H/m). The critical wavelength and the critical magnetic field intensity for infinitesimal ζ_0 are calculated as $\lambda_{\mathrm{CL}}= 2\pi/k_{\mathrm{CL}}=1.02\times10^{-2}$ m and $H_{\mathrm{CL}}=7.31, 3.75, 2.57, 1.98\,[\times10^4\mathrm{A/m}]$ (μ_1/μ_0=1.1, 1.2, 1.3, 1.4). Applied magnetic field is vertical and homogeneous, and its intensity is in the range of $0.95H_{\mathrm{CL}}\leq H_0\leq 1.03H_{\mathrm{CL}}$. The interface profile as Fig.1(a) is composed of three basic modes with the wavenumber vector $\boldsymbol{k}_\mu=m_\mu\boldsymbol{A}+n_\mu\boldsymbol{B}$, $(m_\mu,\,n_\mu)=(2,0),\,(0,2),\,(2,2)$ as shown in Fig.1(b) by open circles, where $\boldsymbol{A}$, $\boldsymbol{B}$ are the basic vectors crossing each other with the angle of 120°. The amplitude of these modes are same, and in the range of $0.10\,\mathrm{mm}\leq\zeta_0\leq 0.50\,\mathrm{mm}$. The dispersion relation for linear wave is applicable for two-dimensional interface when the size of all wavenumbers are same as $|\boldsymbol{k}_\mu|=k$. In Fig.1(b), shaded squares show the value of $h(k)$ at $H_0=H_{\mathrm{CL}}$ for infinitesimal ζ_0, as well as the arc representing the position of the critical wavenumber $k=k_{\mathrm{CL}}$. Modes as many as 18×18 in the k-space cover those generated by the nonlinearity of the surface tension and the magnetic stress difference.

Figure 1(c) shows the bifurcation diagram obtained from the behavior of $h(k)$ as explained at the end of the previous section. While ζ_0 is less than 0.40 mm, G and C in $h(k)$, proportional to k and k^3 for infinitesimal ζ_0, respectively, are positive, but T proportional to k^2 is negative. As long as C is predominant more than T in the higher domain of k, the minimum of $h(k)$ necessarily exists as shown by open circles in Fig.2(b), whose value decreases from positive to negative as H_0 increases. Then, we can determine the critical magnetic field intensity H_{c} and the critical wavenumber k_{c} as defined from the condition that h=0 and $\partial h/\partial k$=0. In Fig.1(c), H_{c}'s thus obtained are shown by closed symbols.

When ζ_0 is larger, $h(k)$ at higher k begins to decrease monotonically (Fig.2(c)) due to the modes generated by the nonlinearity in T more than in C, and the unstable domain with $h(k)<0$ appears in k. Since we find no minimum on $h(k)$ in this case, we use the applied magnetic field intensity H_0 when $h(k)$=0 at $k=k_{\mathrm{CL}}$ as the measure for instability, though $\partial h/\partial k|_{k=k_{\mathrm{CL}}}=0$ is not satisfied, of course. Open symbols in Fig.1(c) show such magnetic field intensities.

For smaller ζ_0, the bifurcation is **subcritical** at any relative permeability μ_1/μ_0 since $H_{\mathrm{c}}<H_{\mathrm{CL}}$. For larger μ_1/μ_0, the critical magnetic field intensity H_{c} decreases since the effect of magnetic field is enhanced. The behavior of the branch line in Fig.1(c), especially at large ζ_0 or its dependence on μ_1/μ_0, differs from that predicted by [4], which should be investigated intensively in the future.

References

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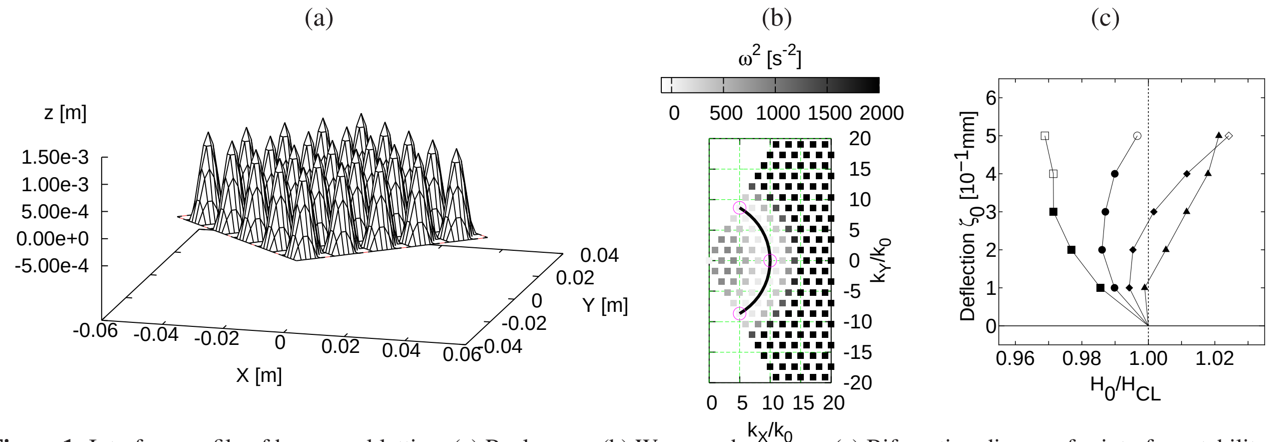

Figure 1. Interface profile of hexagonal lattice. (a) Real space, (b) Wavenumber space, (c) Bifurcation diagram for interface stability: μ_1/μ_0=1.1 (▲△), 1.2 (◆◇), 1.3 (●○), 1.4 (■□).

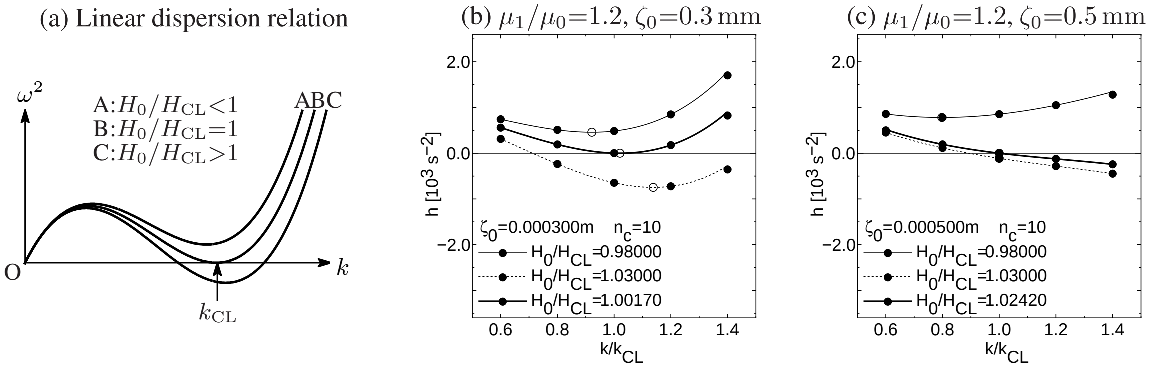

Figure 2. Nonlinear response $h(k)$ depending on μ_1/μ_0, ζ_0 and H_0.