DIFFUSIVE EFFECTS ON ACOUSTIC WAVE PROPAGATION IN A GAS-FILLED CHANNEL SUBJECT TO TEMPERATURE GRADIENT

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Summary This paper examines diffusive effects due to viscosity and heat conduction on acoustic wave propagation in a gas-filled channel subject to temperature gradient axially. Within a linear theory based on a narrow-tube approximation, the system of fluid-dynamical equations is reduced to a thermoacoustic-wave equation for excess pressure. It is shown that the diffusive effects affect wave propagation through shear stress and heat flux on wall surfaces, which are given by memory integrals respect to the pressure gradients.

BACKGROUND

Study on diffusive effects due to viscosity and heat conduction in acoustic wave propagation through a gas enclosed in a channel or a tube has a long history since the pioneering work by Helmholtz and Kirchhoff. Because thermal processes in acoustics are usually assumed to be adiabatic, the effects have been treated as being secondary. But when waves are propagated in a narrow channel or tube, they are subject significantly to diffusive effects. Lord Rayleigh showed that the waves are then diffused too easily to be propagated therein. Thus such a case has attracted less attention so far.
But if temperature gradient is present along the channel or tube, the situation changes dramatically. It happens that a gas becomes unstable due to diffusive effects so that oscillations of gas occur spontaneously. In fact, heat engines exploiting such phenomena have been devised [1,2]. Because of temperature gradient, heat flow exists in background even if the gas is quiescent under uniform pressure. The gas is then in non-equilibrium thermally. In analogy with stability of flow, it is expected that disturbances grow by gaining energy from the heat flow to destabilize the gas, giving rise to wave propagation eventually. A behavior of gas in such a situation has not yet been examined generally. This motivates us to consider wave propagation in a gas-filled channel to examine diffusive effects under temperature gradient.

SET-UP OF THE PROBLEM

Consider a gas-filled channel, as shown in Figure 1, of span width $2H$ enclosed by two flat plates, which extends infinitely in both directions. No variations in the direction to perpendicular to the sheet are considered. Let the gas be quiescent under uniform pressure p_0, where no gravity is assumed. Taking the x-axis along the channel, wall temperature, denoted by $T_e(x)$, is assumed to vary in the x direction. If temperature gradient is gentle over an axial length comparable with the span width, gas temperature in quiescent state is regarded as being equal to wall temperature.
Assuming infinitesimally small disturbances, equation of continuity, Navier-Stokes equation, equation of energy and equation of state for the ideal gas are linearized about the quiescent state. Here a narrow-tube approximation is employed, which stipulates that

(i) a typical axial length in the temperature distribution is long enough in comparison with H,
(ii) a typical axial wavelength of disturbances is also long enough in comparison with H,
(iii) a typical thickness of viscous and thermal diffusion layers is not restricted in comparison with H.

This assumption leads to equations equivalent to those derived by a boundary-layer approximation. It then follows immediately that the excess pressure p' over p_0 is uniform over the span width.

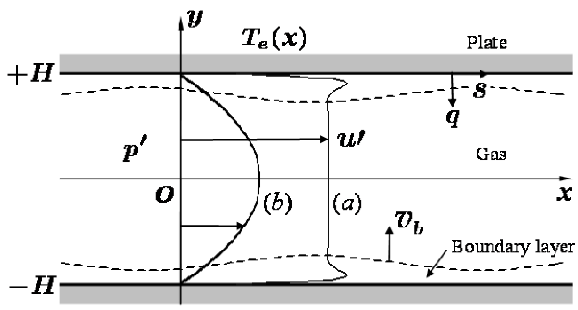

Figure 1. Illustration of a gas-filled channel where (a) and (b) represent instantaneous profiles of axial velocity u' in cases that diffusion layer is thin or thick, respectively.

RESULTS

Using this result for the pressure, the linearized system of equations is reduced to a following equation [3]:

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial}{\partial x}\left(a_e^2 \frac{\partial p'}{\partial x}\right) = -\frac{\partial}{\partial x}\left(a_e^2 \frac{s}{2H}\right) + \frac{a_e^2}{c_p T_e}\frac{\partial}{\partial t}\left(\frac{q}{2H}\right), \tag{1}$$

t being the time, where $a_e(x)$ denotes a local adiabatic sound speed given by $\sqrt{\gamma p_0 / \rho_e}$, γ and $\rho_e(x)$ being the ratio of specific heats and the density of gas in the quiescent state with $\rho_e T_e$ constant; s and q denote, respectively, shear stress

acting on the gas and heat flux flown into the gas at both wall surfaces, c_p being a specific heat at constant pressure. As a_e varies with temperature, the left-hand side represents lossless propagation in gas non-uniform in temperature. The right-hand side represents the diffusive effects due to shear stress and heat flux at the wall surfaces, which appear in the form of acoustic dipole and monopole, respectively.
Here explicit expressions of the shear stress and the heat flux are available in terms of p'. The shear stress is expressed as

$$s = 2\sqrt{\nu_e}\mathcal{M}\left(\frac{\partial p'}{\partial x}\right), \tag{2}$$

$\nu_e(x)$ being a kinematic viscosity of gas, while the time derivative of the heat flux is expressed as

$$\frac{\partial q}{\partial t} = -2c_p T_e\sqrt{\nu_e}\left\{\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\mathcal{M}_P\left(\frac{1}{a_e^2}\frac{\partial^2 p'}{\partial t^2}\right) - \frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\left[\frac{1}{1-\mathrm{Pr}}\mathcal{M}\left(\frac{\partial p'}{\partial x}\right) - \frac{1}{(1-\mathrm{Pr})\sqrt{\mathrm{Pr}}}\mathcal{M}_P\left(\frac{\partial p'}{\partial x}\right)\right]\right\}, \tag{3}$$

where $\mathcal{M}_P$ is a functional (integral) of $\partial p'/\partial x$ defined by

$$\mathcal{M}_P\left[\frac{\partial p'(x,t)}{\partial x}\right] \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{G[\nu_e(t-\tau)/\mathrm{Pr}\,H^2]}{\sqrt{t-\tau}}\frac{\partial p'(x,\tau)}{\partial x}\mathrm{d}\tau \qquad \text{with} \qquad G(t) = 1 + 2\sum_{n=1}^{\infty}(-1)^n\exp\left(-\frac{n^2}{t}\right), \tag{4}$$

and $\mathcal{M}$ is defined by setting the Prandtl number Pr in $\mathcal{M}_P$ formally equal to unity. Both the shear stress and the heat flux are expressed in memory integrals of the pressure gradient. It is noted that the shear stress affects the heat flux through $\mathcal{M}$ if the temperature gradient is present.
Substituting these into (1), the wave equation for p' is derived as follows:

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial}{\partial x}\left(a_e^2\frac{\partial p'}{\partial x}\right) + \frac{\partial}{\partial x}\left[\frac{a_e^2\sqrt{\nu_e}}{H}\mathcal{M}\left(\frac{\partial p'}{\partial x}\right)\right] + \frac{\sqrt{\nu_e}}{H}\left\{\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\mathcal{M}_P\left(\frac{\partial^2 p'}{\partial t^2}\right) - \frac{a_e^2}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\left[\frac{1}{1-\mathrm{Pr}}\mathcal{M}\left(\frac{\partial p'}{\partial x}\right) - \frac{1}{(1-\mathrm{Pr})\sqrt{\mathrm{Pr}}}\mathcal{M}_P\left(\frac{\partial p'}{\partial x}\right)\right]\right\} = 0. \tag{5}$$

This integrodifferential equation is a thermoacoustic-wave equation valid generally in linear regime [3].
A kernel function $G(t)/\sqrt{\pi t}$ involved in the integrals is a kind of relaxation function. It is found that this is well approximated as $1/\sqrt{\pi t}$ for $0 < t < M \approx 0.213$, while $2\exp(-\pi^2 t/4)$ for $M < t$. The memory effects appear greatly for a short time but decay exponentially in the course of time.

DISCUSSIONS

Discussions are made on approximations of the thermoacoustic-wave equation for a short-time behavior and a long-time one in comparison with a typical diffusion time H^2/ν_e. These approximations correspond to cases where the diffusion layer is thin or thick, respectively, compared with the span width. In the former case, it may occur that the diffusion layer can *do work* on the gas outside to destabilize it, depending on the magnitude of temperature gradient. In the latter case, it is revealed that the temperature gradient can *cause wave propagation* by combined action of diffusive effects, and that negative diffusion may occur.
The heat conduction so far concerned is due to the one of gas and no effects of heat conduction in solid plates have been taken into account. As long as the heat capacity of the plates per unit axial length of the channel is much larger than that of the gas, or the thermal conductivity of the solid is much larger than that of the gas, temperature variations in the plates may be negligible. But if they are regarded as being finite, the variations must be taken into account. Such a case occurs when wave propagation takes place in gas-filled channels enclosed in a stack of plates [2] (see Figure 2). Of course, the diffusive effects due to the heat conduction of solid modify thermoacoustic-wave equation [4]. It is unveiled that although the effects may be usually negligible, they become enhanced *selectively* when the ratio of the plate thickness $2d$ to the span width $2H$ takes special values.

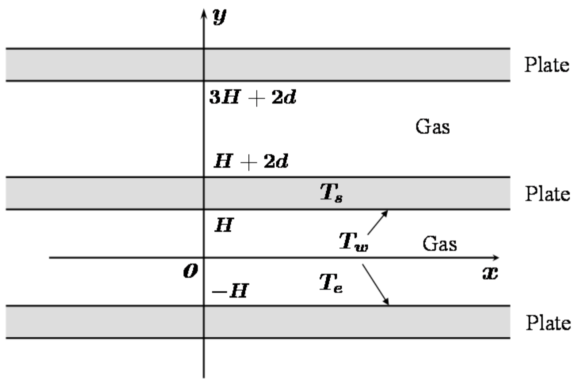

Figure 2. Illustration of gas-filled channels consisting of a stack of solid plates of thickness $2d$ where T_s and T_w denote, respectively, the temperature of solid plate and the one of wall surface.

References
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