LONG WAVE GENERATION ABOVE A CYLINDRICAL SILL

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Summary Waves generated in a two-horizontal-dimension fluid domain of uniform depth by a deformation of the bottom boundary was elegantly studied by Hammack [3]. In the current paper we investigate the effect of bathymetry by analytically solving the forced Linear Shallow Water Equations (LSWE) when the bathymetry consists of a cylindrical sill sitting on a flat bottom. We use Hammack's seafloor deformation and we follow a similar approach as Renzi & Sammarco [4].

WAVE DESCRIPTION

The bathymetry consists of a cylinder of radius r_c^* and height h_c^* (starred quantities are dimensional), sitting on an indefinite horizontal surface. The depth when $r^* < r_c^*$ is h_1^* while otherwise it is h_2^* . The seafloor is vertically uniformly displaced at the circular area defined by $r_0^* > r_c^*$. We use the following non-dimensionalization:

$$r = r^*/r_c^* \ , \quad t = \sqrt{\frac{g}{r_c^*}}t^* \ , \quad \zeta = \zeta^*/f_0^* \ , \quad f = f^*/f_0^* \ , \quad h = h^*/h_c^* \tag{1}$$

where f_0^* is the maximum vertical displacement of the seafloor. The forcing term is $f(r,t) = (1-e^{-\gamma t})\mathcal{H}(r_0 - r)$, $t > 0$ with $r_0 > 1$, $\mathcal{H}$ the Heaviside function and γ a coefficient that defines the speed of the displacement. The forced LSWE in polar coordinates (r, θ) is

$$\zeta_{tt} - h\left(\frac{1}{r}\zeta_r + \zeta_{rr} + \frac{1}{r^2}\zeta_{\theta\theta}\right) - h_r\zeta_r = f_{tt} \tag{2}$$

where $\zeta(r, \theta, t)$ is the free-surface elevation, t is the time and $h(r)$ is the bottom depth with respect to the undisturbed free-surface $z = 0$. We employ the Laplace transform with complex ω being the non-dimensional transform parameter. The transformed Eqn. (2) yields

$$r^2\hat{\zeta}_{rr} + r\left(1 + r\frac{h_r}{h}\right)\hat{\zeta}_r - \frac{\omega^2 r^2}{h}\hat{\zeta} + \hat{\zeta}_{\theta\theta} = -\frac{r^2}{h}\hat{f}_{tt}, \quad \hat{f}_{tt} = \int_0^\infty f_{tt}(r,\theta,t)e^{-\omega t}\mathrm{d}t \tag{3}$$

The near field ($r^* < r_c^*$; $r < 1$) **in the transformed space**
Since in the near field $h = h_1^*/h_c^* \doteq c_d$ and everything is axisymmetric, Eqn. (3) becomes

$$r^2\hat{\zeta}_{rr} + r\hat{\zeta}_r - \frac{\omega^2 r^2}{c_d}\hat{\zeta} = -\frac{r^2}{c_d}\hat{f}_{tt} \tag{4}$$

If we take the homogeneous form of Eqn. (4) (by setting zero the right hand side) it is straightforward to see that the above equation becomes a standard modified Bessel equation of zero-th order, whose solutions are the two modified Bessel functions I_0 and K_0. The general solution of the homogeneous equation has a branch-cut on the negative real axis of the complex plane ω introduced by K_0. Boundedness of the free-surface elevation at $x = 0$ requires the solution to depend only on I_0. In order to find the solution of the inhomogeneous equation (4) we employ the method of variation of parameters. Hence the solution of the forced ordinary differential equation (4) is

$$\hat{\zeta}(r;\omega) = \alpha_1 \, I_0\left(\frac{\omega r}{\sqrt{c_d}}\right) - P(r;\omega) \tag{5}$$

$$\text{where} \ \ P(r;\omega) = \int_0^r \frac{\hat{f}_{tt}(\rho;\omega)}{c_d W_1(\rho;\omega)}\left[I_0\left(\frac{\omega\rho}{\sqrt{c_d}}\right) K_0\left(\frac{\omega r}{\sqrt{c_d}}\right) - I_0\left(\frac{\omega r}{\sqrt{c_d}}\right) K_0\left(\frac{\omega\rho}{\sqrt{c_d}}\right)\right] \mathrm{d}\rho \tag{6}$$

is the particular solution and $W_1(\rho;\omega) = \sqrt{c_d}/\omega\rho$ is the Wronskian [1] of the two homogeneous solutions.

The far field ($r^* > r_c^*$; $r > 1$) **in the transformed space**
In the far field $h = h_2^*/h_c^* = 1 + c_d \doteq k_d$. By performing the same kind of analysis as for the near field, one can show that

$$\hat{\zeta}(r;\omega) = \beta_2 \, K_0\left(\frac{\omega r}{\sqrt{k_d}}\right) - \Pi(r;\omega) \tag{7}$$

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$$\text{where } \Pi(r;\omega) = \int_1^r \frac{\hat{f}_{tt}(\rho;\omega)}{k_d W_2(\rho;\omega)} \left[I_0\left(\frac{\omega\rho}{\sqrt{k_d}}\right) K_0\left(\frac{\omega r}{\sqrt{k_d}}\right) - I_0\left(\frac{\omega r}{\sqrt{k_d}}\right) K_0\left(\frac{\omega\rho}{\sqrt{k_d}}\right)\right] \mathrm{d}\rho \tag{8}$$

is the particular solution and $W_2(\rho;\omega) = \sqrt{k_d}/\omega\rho$ is the Wronskian [1].

Matching at $r = 1$

By matching the free-surface elevation and the radial fluxes at $r = 1$ we obtain the expressions for α_1 and β_2 :

$$\alpha_1(\omega) = \frac{\hat{f}_{tt}}{\sqrt{c_d}} \left[\frac{1}{\sqrt{c_d}} \frac{W\left(K_0\left(\frac{\omega}{\sqrt{c_d}}\right) \ , \ K_0\left(\frac{\omega}{\sqrt{k_d}}\right)\right)}{W\left(I_0\left(\frac{\omega}{\sqrt{c_d}}\right) \ , \ K_0\left(\frac{\omega}{\sqrt{k_d}}\right)\right)} I_1\left(\frac{\omega}{\sqrt{c_d}}\right) - \frac{1}{\omega} + \frac{1}{\sqrt{c_d}} K_1\left(\frac{\omega}{\sqrt{c_d}}\right)\right]$$

$$\beta_2(\omega) = \frac{1}{\omega\sqrt{c_d}} \frac{\hat{f}_{tt} \, I_1\left(\frac{\omega}{\sqrt{c_d}}\right)}{W\left(K_0\left(\frac{\omega}{\sqrt{k_d}}\right) \ , \ I_0\left(\frac{\omega}{\sqrt{c_d}}\right)\right)} \tag{9}$$

where W denotes the Wronskian. By employing the inverse Laplace transform we obtain the free-surface elevation in the physical space.

The near field ($r < 1$) in physical space

$$\begin{aligned}
\zeta(r;t) \quad = \quad & \frac{1}{2\pi} \int_0^\infty \left\{ \frac{1}{ic_d} \frac{\gamma^2}{\gamma - i\omega} \frac{W\left(\frac{i\pi}{2} H_0\left(\frac{\omega}{\sqrt{c_d}}\right) \ , \ \frac{i\pi}{2} H_0\left(\frac{\omega}{\sqrt{k_d}}\right)\right)}{W\left(J_0\left(\frac{\omega}{\sqrt{c_d}}\right) \ , \ \frac{i\pi}{2} H_0\left(\frac{\omega}{\sqrt{k_d}}\right)\right)} J_1\left(\frac{\omega}{\sqrt{c_d}}\right) \right. \\
& \left. + \frac{\gamma^2}{\gamma - i\omega} \frac{1}{i\omega\sqrt{c_d}} \left(1 - \frac{i\omega\pi}{2\sqrt{c_d}} H_1\left(\frac{\omega}{\sqrt{c_d}}\right)\right)\right\} J_0\left(\frac{\omega r}{\sqrt{c_d}}\right) e^{-i\omega t} \mathrm{d}\omega + c.c. \\
& -\frac{1}{4} \int_0^r \int_0^\infty \frac{\gamma^2}{\gamma - i\omega} \frac{\omega\rho}{(c_d)^{3/2}} \left[J_0\left(\frac{\omega\rho}{\sqrt{c_d}}\right) H_0\left(\frac{\omega r}{\sqrt{c_d}}\right) - J_0\left(\frac{\omega r}{\sqrt{c_d}}\right) H_0\left(\frac{\omega\rho}{\sqrt{c_d}}\right)\right] e^{-i\omega t} \, \mathrm{d}\omega \, \mathrm{d}\rho - c.c.
\end{aligned} \tag{10}$$

Where J_n is the Bessel function of first kind and order n, while H_n is the Hankel function of first kind and order n. C.c denotes the complex conjugate of the integral over ω.

The far field ($r > 1$) in physical space

$$\begin{aligned}
\zeta(r;t) = & -\frac{1}{4} \int_0^\infty \frac{\gamma^2}{\gamma - i\omega} \frac{1}{i\omega\sqrt{c_d}} \frac{J_1\left(\frac{\omega}{\sqrt{c_d}}\right) H_0\left(\frac{\omega r}{\sqrt{c_d}}\right)}{W\left(\frac{i\pi}{2} H_0\left(\frac{\omega}{\sqrt{k_d}}\right) \ , \ J_0\left(\frac{\omega}{\sqrt{c_d}}\right)\right)} e^{-i\omega t} \mathrm{d}\omega - c.c. \\
& -\frac{1}{4} \int_1^r \int_0^\infty \frac{\omega\rho}{(k_d)^{3/2}} \frac{\gamma^2}{\gamma - i\omega} \mathcal{H}(r_0 - r) \left[J_0\left(\frac{\omega\rho}{\sqrt{k_d}}\right) H_0\left(\frac{\omega r}{\sqrt{k_d}}\right) - J_0\left(\frac{\omega r}{\sqrt{k_d}}\right) H_0\left(\frac{\omega\rho}{\sqrt{k_d}}\right)\right] e^{-i\omega t} \, \mathrm{d}\omega \, \mathrm{d}\rho - c.c.
\end{aligned} \tag{11}$$

DISCUSSION

The evolution of the free-surface elevation is analytically derived in terms of Bessel and Hankel functions. It is clear that a significant parameter is c_d which depends on the height of the sill and we are currently exploring its importance. The analytical solution will be compared to the solution derived by Hammack [3], when the seafloor is flat as well as to the numerical results of a nonlinear shallow water equations solver [2].

References

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