DIRECT ENERGY TRANSFORM TO CROSS-WAVES IN A RECTANGULAR CHANNEL OF FINITE DIMENSIONS

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Summary The phenomenon of a cross-wave generation in a free surface of a fluid confined in a rectangular tank with the finite dimensions and one wall as a flap wavemaker is investigated. Cross-waves are studied experimentally and theoretically using Lame's method of superposition. This method allows to construct a simple mathematical model, which shows how the cross-waves can be generated directly by the wavemaker motion without taking into account the presence of any axisymmetric waves.

INTRODUCTION

It is well known that vibrating bodies immersed in a fluid may generate various types of wave patterns at the free surface of the fluid. An example is the formation of cross-waves, which were first found and described by Faraday (1831). Cross-waves have crests perpendicular to the wavemaker (hence their name), and their frequency is half that of the wavemaker.
Cross-waves have been the subject of many studies during the last 40 years. It was Garrett (1970) who first showed how energy is transferred from the wavemaker to the cross-wave in a mathematical model including a mean motion of the free surface. However, this primary motion of the free surface is not sufficient to supply the energy to the cross-waves in the long tank,for example. Therefore, the cross-waves must derive their energy in some way directly from the wavemaker. In the previous theoretical studies (for a detailed survey, see Miles & Henderson (1990)) authors used a modification of Havelock's (1929) solution of the wavemaker problem for a semi-infinite tank with an infinite depth or length and a radiation condition instead of the zero velocity condition at the finite nonmoving boundaries and their solution contains all the information about the behaviour of axisymmetric modes along the tank and the influence of these modes on the stability of the resonant cross-wave. We use a different method for solving the wavemaker problem, applying Lamé's (1852) superposition method. This method allows one to construct a simpler mathematical model, which shows how the cross-wave can be generated directly by the wavemaker motion without having to take into account the presence of any axisymmetric waves at the free surface. This simpler mathematical model of the excitation of the resonant cross-waves may be the easiest way to understand energy transform from the wavemaker directly to the motion of cross-waves. For the first time this method of solution was used in the papers Krasnopolskaya & van Heijst (1996)for excitation of cross-waves in an annular fluid tank with a radially vibrating inner cylinder. In addition to this theoretical analysis we have carried out laboratory experiments. The experimental observations agree very well with the theoretical results.

THEORETICAL ANALYSIS

We consider cross-waves in a rectangular tank. Assuming that the fluid is inviscid and incompressible, and that the induced motion is irrotational, we consider cross-waves in a rectangular tank with the length L, the width b and the depth h at $z = -h$. The wavemaker vibrations as a flap (at the boundary when $x = 0$) are described by the function

$$u(z,t) = F(z)\sin(\omega t) = \left(a + \frac{a}{h}z\right)\sin(\omega t), \tag{1}$$

where a is the amplitude of vibrations; ω is the frequency.
The velocity field can be written as $\vec{v} = \vec{\nabla}\varphi$ with $\varphi(x, y, z, t)$ the velocity potential. The governing equation is

$$\nabla^2\varphi = 0 \,\text{at}\, F(z)\,\cos(\omega t)\, \le x \le L,\ 0 \le y \le b,\ -h \le z \le \xi(x, y, t), \tag{2}$$

where $\xi(x, y, t)$is the free surface elevation.
The dynamic and kinematic boundary conditions in the free surface can be written as

$$\varphi_t + \frac{1}{2}\left(\vec{\nabla}\varphi\right)^2 + \mathrm{g}\xi = F_0(t) \text{ at } z = \xi\,(x, y, t)\,; \tag{3}$$

$$\varphi_z = \vec{\nabla}\varphi \cdot \vec{\nabla}\xi + \xi_t \text{ at } z = \xi\,(x, y, t)\,, \tag{4}$$

with g the gravitational acceleration. Here and later the subscripts x, y, z, t signify partial differentiation.

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The normal velocity vanishes at the solid flow boundaries and the main problem is to correctly use the kinematic condition at the vibrating wavemaker:

$$\varphi_x = \omega F(z)\cos\omega t + F'(z)\,\varphi_z \sin\omega t \quad \text{at} \quad x = F(z)\sin\omega t. \tag{5}$$

In order to obtain a solution of the problem we use the analytical method of superposition. The idea of the superposition method was first proposed by Lamé (1852) in his classical lectures on the theory of elasticity. According to this superposition method, the potential can be written as the sum of three harmonic functions $\varphi = \varphi_0+\varphi_1+\varphi_2$. The potential ϕ_0 governed by the axisymmetric boundary problem is $\varphi_0 = -\dot{\tilde{F}}_0(t)\frac{(x-L)^2}{2L}+\dot{\xi}_0\frac{(z+h)^2}{2h}$ We here consider the wave pattern that is realized during parametric resonance, when the excitation frequency ω is twice as large as the eigenfrequencies ω_{nm}, namely: $\omega \approx 2\omega_{nm}$. So we consider a resonance case in which the free surface displacement can be approximated only by the resonance mode and the mean level displacement $\xi \approx \xi_{nm}(t)\cos\dfrac{n\pi x}{L}\cos\dfrac{m\pi y}{b} + \xi_{00}$. And $\xi_{00}(t)$ can be written as $\xi_{00}(t) = \dfrac{ah}{2L}\sin\omega t = \varepsilon_1\,\dfrac{\mathrm{g}}{2\omega_{nm}^2}\,\dfrac{h}{L}\sin\omega t$. Then

$$\varphi_1 \approx \varphi_{n\,m}(t)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{b}\,\frac{\cosh[k_{nm}(z+h)]}{\cosh(k_{n,m}h)}. \tag{6}$$

After applying the boundary conditions we obtain φ_2 in the form

$$\begin{aligned}\varphi_2 = {}&\varepsilon_1\cos\omega t\,\tilde{\Phi}_{00} + \varepsilon_1\cos\omega t\sum_{j=1}^{\infty}\tilde{\Phi}_{0j}\,\cos\frac{j\pi z}{h}\,\frac{\cosh\alpha_{0j}(x-L)}{\cosh\alpha_{0j}} + \\ &+\varepsilon_1\,\dot{\xi}_{nm}(t)\,\sin\omega t\,\tilde{\Phi}_{m0}\,\cos\frac{n\pi y}{b}\,\frac{\cosh\alpha_{m0}(x-L)}{\cosh\alpha_{m0}} + \\ &+\varepsilon_1\,\dot{\xi}_{nm}(t)\,\sin\omega t\sum_{j=1}^{\infty}\tilde{\Phi}_{mj}\,\cos\frac{n\pi y}{b}\,\frac{\cosh\alpha_{mj}(x-L)}{\cosh\alpha_{mj}}.\end{aligned} \tag{7}$$

The two last sums determine the fraction of energy transmitted by the vibrating wavemaker into the cross-waves directly. For the resonant amplitude we should obtain the equation:

$$\begin{aligned}&\ddot{\xi}_{nm} + \omega_{nm}^2\,\xi_{nm} - \frac{9}{16}\,\omega_{nm}^2\,k_{nm}^2\,\xi_{nm}^3 + \frac{3}{4}\,k_{\,nm}^{\,2}\,\xi_{nm}\,\dot{\xi}_{nm}^2 + \\ &+\varepsilon_1\,D_3\,\dot{\xi}_{nm}\,\cos\omega t - \varepsilon_1\,D_4\,\xi_{nm}\,\sin\omega t = 0,\end{aligned} \tag{8}$$

where

$$D_4 = \omega_{nm}^2\,D_1 - k_{nm}\tanh k_{nm}h\,D\,\omega.$$

This equation is typical for the parametric resonance and clearly shows the possibility of parametric resonance for the selected mode. From this equation we may construct evolutionary equations for the amplitude of the resonant dominant mode, study stability problem and determine how much energy transfer from wavemaker motion directly to this dominant mode.

CONCLUSIONS

Cross-waves are studied experimentally and theoretically using Lame's method of superposition. This method allows to construct a simple mathematical model, which shows how the cross-waves can be generated directly by the wavemaker motion without taking into account the presence of any axisymmetric waves. All previous theories have considered the cross-waves excitation task in a rectangular tank applying the Havelock's (1929) solution of the wavemaker problem for a semi-infinite tank with an infinite depth or length and a radiation condition instead of the zero velocity condition at the finite nonmoving boundaries.

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