INTERACTION OF RESONANT GRAVITY WAVES IN WATER OF FINITE DEPTH

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Summary We investigate a fully developed wave system consisted of two traveling primary waves in finite water depth when Phillips' criterion of wave resonance is satisfied. An analytic technique for highly nonlinear equations, namely the homotopy analysis method (HAM), is applied to gain convergent series approximations. It is found that the fully developed wave system has multiple solutions, and the amplitude of the resonant component may be much smaller than the primary ones. This result gives something different from the traditional ideas about resonance, and is of benefit to enrich our understanding about gravity wave resonance.

INTRODUCTION

Phillips [1] pointed out the resonant criterion of a quartet waves in deep water. When two of the four waves have the same wave number, the corresponding resonant criterion in water of finite depth reads:

$$(2\bar{\sigma}_1 - \bar{\sigma}_2)^2 = g|2\mathbf{k}_1 - \mathbf{k}_2| \tanh\left(|2\mathbf{k}_1 - \mathbf{k}_2|d\right), \tag{1}$$

where $\bar{\sigma}_i = \sqrt{gk_i \tanh(k_i d)}$ $(i = 1, 2)$ is the angular frequency of the primary wave in the linear theory, $\mathbf{k}_i$ denotes the wave number, $k_i = |\mathbf{k}_i|$, d is the water depth, and g is the gravity acceleration, respectively. Phillips [1] found that the amplitude of the resonant component would grow with time in the initial status of the interaction, and this is also verified by Longuet-Higgins [2] after carefully calculation. However, following the work of Benney [3] which gave the evolution equations of the wave amplitude, Bretherton [4] found that the wave energy is distributed among oscillation members in a periodic manner after solving the evolution equations over a characteristic time. So the amplitude of the resonant component in the fully developed wave system can not increase infinitely, and the wave system may act in a periodic manner. Recently, Madsen [5] got a third-order solution for bichromatic bi-directional water waves with an ambient current in finite depth using a periodic solution form. In this paper, with the help of the homotopy analysis method (HAM [6,7]), we consider a fully developed wave system consisted of two traveling primary waves in finite water depth when Phillips' criterion of wave resonance is satisfied.

MATHEMATICAL DESCRIPTION

We consider two primary gravity waves with small amplitudes, propagating in water of finite depth. Assumed that the fluid is inviscid, incompressible and the flow is irrotational. The coordinate system (x, y, z) is set on the free surface, with z-axis positive vertically from the free surface. Let d denote the water depth. The governing equation of the velocity potential $\phi(x, y, z, t)$ reads

$$\nabla^2\phi = 0, \quad -d < z < \eta(x, y, t), \tag{2}$$

subject to the two boundary conditions on the unknown free surface $z = \eta(x, y, t)$:

$$\frac{\partial^2\phi}{\partial t^2} \quad + \quad g\frac{\partial\phi}{\partial z} + \frac{\partial|\nabla\phi|^2}{\partial t} + \nabla\phi\cdot\nabla(\frac{1}{2}|\nabla\phi|^2) = 0, \quad g\eta + \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 = 0, \tag{3}$$

and the bottom boundary condition

$$\frac{\partial\phi}{\partial z} = 0, \qquad at \quad z = -d, \tag{4}$$

where

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{5}$$

The above equations are solved by means of the HAM. For details, please refer to Liao [8].

MAIN RESULTS

Without loss of generality, we consider the following case:

$$\frac{\sigma_1}{\bar{\sigma}_1} = \frac{\sigma_2}{\bar{\sigma}_2} = \epsilon, \alpha_1 = 0, \alpha_2 = \frac{\pi}{36}, k_2 = \frac{\pi}{5}, d = 3, \tag{6}$$

where σ_i $(i = 1, 2)$ is the frequency of the two primary waves, and ϵ is chosen to be 1.0003, since Phillips' criterion is valid only for small amplitude waves. According to Phillips' resonance criterion, there are two values of k_1 in the domain $0.4 < k_1 < 1$ when

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the resonance occurs in the case of (6): for each k_1 of them, three solutions with quite different resonant component are found. The energy distributions of the primary and resonant components are shown in Table 1 for the case of $k_1 < k_2$, where

$$\Pi_0 \quad = \quad a_{1,0}^2 + a_{0,1}^2 + a_{2,-1}^2, \qquad \Pi = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} a_{m,n}^2. \tag{7}$$

The solutions of Group 1 in Table 1 show that the amplitude of resonance component ($a_{2,-1}$) is larger (53.45%) than the primary waves ($a_{1,0}$ and $a_{0,1}$). This is in accordance with the traditional idea about resonance. However, in the solutions of Group 3, it is much smaller (10.40%) than the primary ones! This is very interesting. When we change the water depth, there are also three groups of solutions. One of them is shown in Figure 1, where the amplitude of the resonant component is still much smaller than the primary ones (around 10%) even for different water depth.

	$a_{1,0}^2/\Pi$	$a_{0,1}^2/\Pi$	$a_{2,-1}^2/\Pi$	Π_0/Π
Group1	37.37%	9.07%	53.45%	99.89%
Group2	41.67%	36.53%	21.51%	99.72%
Group3	13.51%	76.05%	10.40%	99.95%

Table 1. The energy distribution of three groups solutions in the case of (6) when $\epsilon = 1.0003$, $k_2/k_1 = 1.11165$.

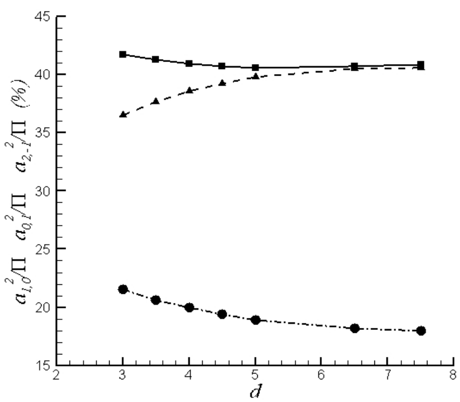

Figure 1. $a_{1,0}^2/\Pi$, $a_{0,1}^2/\Pi$ and $a_{2,-1}^2/\Pi$ versus d when $k_1 < k_2$. Solid line with square: $a_{1,0}^2/\Pi$; Dashed line with delta: $a_{0,1}^2/\Pi$; Dash-dotted line with circle: $a_{2,-1}^2/\Pi$.

Therefore, according to our computations, the resonant wave component may contain maximum *or* minimum wave energy. This result is different from the traditional theory that suggests that the resonant wave should contain most wave energy.

CONCLUSIONS

In the fully-developed wave system consisted of two traveling primary waves in water of finite depth, the amplitude of the resonant component has multiple solutions when Phillips' resonant criterion is satisfied. For some solutions, the resonant component's amplitude is comparable with the primary ones, and this is accordance with the traditional idea. However, in some cases, the amplitude of resonant wave component in finite water depth may be much smaller than the primary ones. Thus, the resonant wave component may contain much less wave energy than the two primary ones. This result gives something different from the traditional ideas about resonance, and is of benefit to enrich our understanding about gravity wave resonance.

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