# PROPAGATION OF INTERFACIAL SOLITARY WATER WAVES IN PRESENCE OF SURFACTANTS

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*Summary* The propagation of long wavelength disturbances on the surface of a fluid layer of finite depth is considered in the presence of a surfactant. The presence of a surfactant affects the surface tension and leads to a surface stress with both tangential and normal components. A large Reynolds number analysis is discussed for an arbitrary surface stress. In this limit the evolution equation for the surface elevation contains contributions from both boundary layers in the fluid; one is adjacent to the free surface while the other lies at the base of the fluid layer. A weakly nonlinear analysis is performed leading to an evolution equation similar to the classic Korteweg-de Vries equation, but modified by additional terms due to the viscosity and to the tangential and normal stress at the surface. When a surfactant is presence, the surface stress depends on the surfactant concentration. Thus the system of equations governing the interfacial wave propagation and the surfactant concentration are solved simultaneously. A parametric study is presented using a pseudo-spectral numerical scheme.

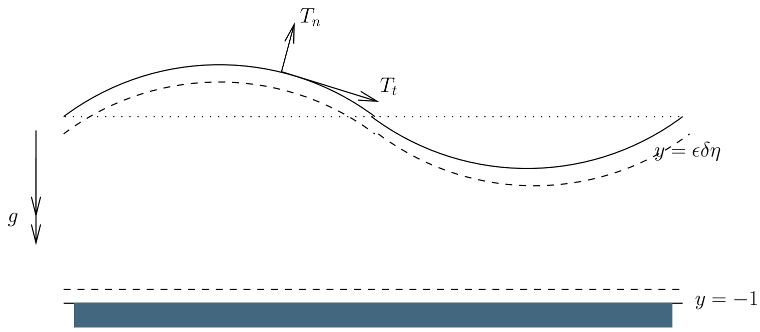
## FORMULATION

A viscous fluid layer, density $\rho$, with average depth $d$ is subjected to a surface stress $\mathbf{T}^* = T_n^*\mathbf{n} + T_t^*\mathbf{t}$, where $(.)^*$ denotes a dimensional quantity and $\mathbf{n}, \mathbf{t}$ are the outward normal and tangent unit vectors at the surface, $\mathbf{x}^* = \mathbf{X}^*$. In this paper we are concerned with the case when surface stress arises due to surface tension effects associated with the presence of a surfactant. Nondimensionalising with $\mathbf{T}^* = \rho g d\mathbf{T}$, gives

$$\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u} + \widehat{\mathbf{y}} = -\nabla p + \frac{1}{R}\nabla^2\mathbf{u}, \qquad R = \frac{\rho d\sqrt{gd}}{\mu}$$

in the fluid layer, with no-slip condition on the lower surface, $y = -1$, and interface conditions

$$\mathbf{n}.\mathbf{X}_t = \mathbf{u}.\mathbf{n}, \qquad p = R^{-1}\mathbf{n}.\mathbf{e}.\mathbf{n} - T_n, \qquad R^{-1}\mathbf{t}.\mathbf{e}.\mathbf{n} = T_t.$$




In this paper we are predominantly concerned with the effect of surface tension on surface wave propagation. If the only surface shear were due to surface tension with constant surface tension parameter $\gamma_0$, then $T_t^* = 0$ and $T_n^* = -\gamma\kappa^*$, where $\kappa^*$ is the dimensional surface curvature. This case has been well studied elsewhere (Hunter & Vanden-Broeck, 1983). The effect of surfactant with concentration $\sigma^*(x,t)$ is that the surface tension parameter, $\gamma$, is reduced from its surfactant free value $\gamma_0$, and hence leads to the occurrence of a tangential shear, given by $T_t^* = \frac{d\gamma}{ds^*}$.

For low surfactant concentrations, we assume that the surface tension parameter varies linearly with surfactant concentration, in which case the non-dimensional surface stress components are given by

$$T_n = -\Gamma_m(1 - E\sigma)\kappa, \qquad T_t = -E\Gamma_m\frac{d\sigma}{ds}, \quad \text{where} \quad E = \frac{\Gamma_0 - \Gamma_m}{\Gamma_m}, \qquad \Gamma_0 = \frac{\gamma_0}{\rho g d^2}, \qquad \Gamma_m = \frac{\gamma_m}{\rho g d^2}, \tag{1}$$

and $\kappa$ is the non-dimensional surface curvature. Here $\Gamma_m$ and $\Gamma_0$ characterise the surface tension forces relative to gravitational forces in the reference state and in the surfactant-free case repectively, and can be considered to be inverse Bond numbers. The factor $E$ characterises the importance of the surfactant for the interfacial dynamics and can be considered either as a Marangoni parameter or as a dimensionless surfactant elasticity.

To complete the system of equations, the surfactant concentration must be related to the fluid flow. In non-dimensional form, with $\sigma^* = \sigma_m^*(1+\sigma)$, the convection-diffusion equation becomes

$$\frac{\partial(H(1+\sigma))}{\partial t} + \frac{\partial(H(1+\sigma)\mathbf{u}.\widehat{\mathbf{x}})}{\partial x} = \frac{1}{\mathrm{Pe}}\frac{\partial}{\partial x}\left(\frac{1}{H}\frac{\partial\sigma}{\partial x}\right), \qquad \mathrm{Pe} = \frac{d\sqrt{gd}}{D_s^*}, \qquad H = \left(1 + {\eta_{x^*}^*}^2\right)^{1/2}. \tag{2}$$

Here $D_s^*$ is the surfactant diffusivity and the Peclet number $\mathrm{Pe}$ characterises the ratio between surfactant advection along the surface and the surfactant diffusion.

In this paper we consider the case when the Reynolds number is large, and relative to the undisturbed fluid depth, the wavelength disturbance is long $O(\epsilon^{-1})$ and the amplitude is small, $O(\delta)$. In the absence of surfactant, this corresponds to the case considered by Hunter & Vanden-Broeck (1983). Assuming first that the Reynolds number, $\mathrm{Re}$ is sufficiently large that $\delta_B = (\epsilon \mathrm{Re})^{-1/2} \ll 1$, boundary layers of thickness $\delta_B$ occur at the base of the fluid and at the surface. The derivation of the general equation governing the surface elevation $\eta(S,T)$ in terms of a prescribed surface stress is discussed in more detail in Hammerton & Bassom (2008). When the surface stress arises due to variable surfactant concentration the governing equation takes the form

$$2\eta_T + 3\eta\eta_S = R_1 + R_2 + R_3 + R_4 + R_5 + R_6, \tag{3}$$

where the terms on the righthandside are given by

$$\begin{gathered} R_1 = -\frac{\varepsilon^2}{\delta}\left(\tfrac{1}{3}\eta_{SSS} + \frac{1}{45}\varepsilon^2\eta^{(v)}\right). \qquad R_2 = \frac{\delta_B}{\delta}\,\mathcal{S}(\eta_S), \qquad R_3 = \frac{\delta_B\epsilon^2}{\delta}\,\mathcal{S}(\eta_{SSS}), \\ R_4 = \frac{\epsilon^2\Gamma_m}{\delta}\eta_{SSS}, \qquad R_5 = \alpha\epsilon^2\delta_B(\widetilde{\sigma}\eta_{SS})_S, \qquad R_6 = \alpha\left(\mathcal{R} - \,\mathcal{S}\left(\widetilde{\sigma}_{ST}\right)\right), \\ \mathcal{R} = -\eta\,\mathcal{S}\left(\widetilde{\sigma}_{SS}\right) - \eta_S\,\mathcal{S}\left(\widetilde{\sigma}_S\right) + \alpha\,\mathcal{S}\left(\widetilde{\sigma}_{SS}\right)\,\mathcal{S}\left(\widetilde{\sigma}_S\right) - \frac{\delta_B}{\delta}\widetilde{\sigma}_S. \end{gathered} \tag{4}$$

where $\alpha = (\Gamma_m - \Gamma_0)/\delta_B$ and the operator $\mathcal{S}(.)$ is the half-derivative.

The term $R_1$ is due to the fluid flow in the bulk of the fluid and arises in the standard derivation of the KdV equation for an inviscid fluid in the absence of surface tension. $R_2$ is the effect of the viscous boundary layer at the rigid base on the fluid. A similar term arises due to boundary-layer dissipation when considering shock-wave propagation in tubes, Sugimoto (1991). $R_3$ is the viscous effect of the boundary layer at the surface in the absence of any applied tangential surface stress. Since this is $O(\epsilon^2)$ smaller than the effect of the boundary layer at the base, this term is dropped from the subsequent analysis. $R_4$ is the effect of the normal shear due to a constant surfactant concentration, and combines with $T_1$ to give the standard dispersion term for a solitary wave with constant surface tension parameter. $R_5$ is the effect of non-uniform surfactant concentration on the normal stress. Since it is uniformly small compared with $R_6$, the effect of tangential surface stress, the term $R_5$ is dropped in the subsequent analysis.

Finally a second equation relating the surfactant concentration and the surface elevation is obtained. The convection-diffusion equation (2) relates the surfactant concentration to the tangential fluid velocity at the surface, while the local solution in the surface boundary layer provides a relation between the fluid velocity and the applied stress. The equation for surfactant concentration then becomes

$$\phi\widetilde{\sigma}_S + \alpha\,\mathcal{S}\left(\widetilde{\sigma}_S\right) + \widetilde{\sigma} = \eta, \qquad \phi = \frac{\epsilon}{\mathrm{Pe}} \tag{5}$$

while the nonlinear wave equation can be written as

$$\frac{d}{dT}\left(2\eta + \alpha\,\mathcal{S}\left(\widetilde{\sigma}_S\right)\right) = -3\eta\eta_S + a\eta_{SSS} - b\eta^{(v)} + q\,\mathcal{S}(\eta_S) + \alpha\mathcal{R}. \tag{6}$$

Thus the system contains four positive dimensionless parameters: $\phi$ an inverse Peclet number, $\alpha$ a Marangoni number (characterising the relative effect of tangential stress, compared to gravitational force), $q$ the effect of viscous dissipation due to the base boundary layer and $b$ the effect of higher order dispersion terms which is only important when $\Gamma_m$ is close to the critical value of $\frac{1}{3}$. The final parameter $a$ is equal to $\pm 1$ depending on whether $\Gamma_m$ is greater or less than $\frac{1}{3}$.

Numerical results are obtained using a pseudospectral scheme. As well as the well-documented advantages of pseudospectral methods for nonlinear wave propagation, an additional advantage here is that the half-derivative terms are evaluated very efficiently. In the absence of viscous dissipation, under certain conditions the presence of surfactant appears to lead to growth in wave amplitude suggestive of a finite-time singularity. However including viscous dissipation counteracts this growth. In the remainder of the paper, a more comprehensive parametric study is presented.

## References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