A POINT VORTEX MODEL OF SINGLY-PERIODIC FOUR-VORTEX WAKE STRUCTURES

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Summary We present a two-dimensional mathematical model consisting of four point vortices in a singly-periodic domain. We use this model as a means to explore the structure and dynamics of laminar wakes that consist of regular groupings of four vortices. The model system is Hamiltonian, and an assumed discrete spatial symmetry makes it integrable. Some initial conditions lead to relative equilibria, in which a vortex configuration moves without change in size or shape, but most configurations evolve dynamically. The model demonstrates a variety of different motions that we classify using the structure of the phase space. Scaled comparisons of the model with experiments show good correspondence, and hence enable classification of the experimental vortex trajectories and estimation of the experimental vortex strengths.

Consider the laminar wakes formed behind bluff bodies that are submerged in a viscous fluid moving with a free stream velocity U. When the bodies oscillate or are in close proximity to one another, the vortices in these wakes can exhibit a variety of patterns. One such pattern is the standard von Kármán street, in which the relative vortex spacing remains essentially constant as the wake progresses downstream. However, in many cases the wake structure is more exotic, and the vortex positions evolve dynamically. If the wake pattern consists of regular groupings of three vortices, a spatially periodic point vortex model has been found to give a qualitative representation of the relative vortex dynamics [1]. In many cases the wake patterns consist of regular groupings of four vortices with net zero circulation [4], such as in the example shown in Fig. 1(a). Until recently [3], these experimental four-vortex wakes had been classified only as belonging to either a '2C mode', in which the two like-signed vortices corotate, or a '2P mode', in which the opposite-signed vortices move as a pair. We present a model, generalized from [3], that suggests a rich dynamics with additional distinctions.

(a)

(b)

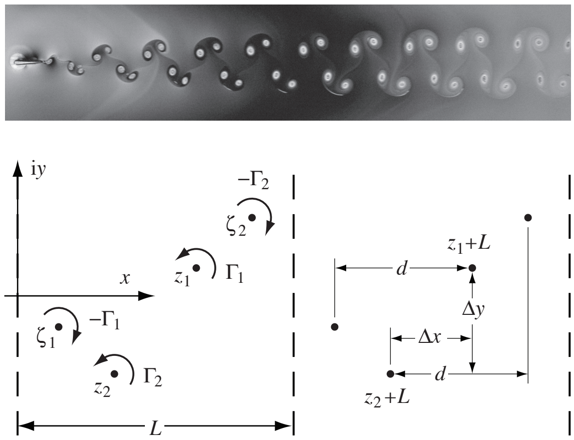

Figure 1. (a) An example of a four-vortex wake generated behind a stationary foil that is forced to flap in a flowing soap film, adapted from [2]. Flow is from left to right. (b) Model representation of a vortex street with four vortices per period.

We consider a two-dimensional potential flow model that assumes a spatially periodic arrangement of four point vortices, as shown in Fig. 1(b). The position of vortex α in the complex plane is given by $z_\alpha = x_\alpha + \mathrm{i}\, y_\alpha$ and its circulation by Γ_α. The equations of motion for these vortices in a periodic domain of width L are well-known and can be found in, e.g., [1]. The general four-vortex system is not integrable, and hence we impose a discrete spatial symmetry to make the system mathematically tractable. Based on the patterns observed in a variety of experiments, including that in Fig. 1(a), we assume the vortex circulations and positions are related via

$$\Gamma_3 = -\Gamma_1, \quad z_3 = z_1^* - d = \zeta_1, \tag{1a}$$
$$\Gamma_4 = -\Gamma_2, \quad z_4 = z_2^* + d = \zeta_2, \tag{1b}$$

where the asterisk denotes complex conjugation. These assumptions generalize our previous results [3] by allowing for $\Gamma_1 \neq \Gamma_2$. In order for these constraints to be maintained by the dynamics of the system it is necessary to have $d = n\,L/2$, where n is an integer. Here we focus on the case $n = 1$.

The periodic point vortex system is Hamiltonian [1]. If we let $z_1 - z_2 = \Delta x + \mathrm{i}\,\Delta y$, then the general Hamiltonian can be reduced to

$$\mathbb{H}(\Delta x, \Delta y) = -\frac{S^2\gamma\,(1-\gamma)}{2\pi}\Bigg\{\ln\left[\frac{\sin^2(\pi\,\Delta x/L) + \sinh^2(\pi\,\Delta y/L)}{\cos^2(\pi\,\Delta x/L) + \sinh^2\big[\wp + (1-2\gamma)\,\pi\,\Delta y/L\big]}\right] - \frac{\gamma}{1-\gamma}\ln\left[\cosh\left(\wp + 2(1-\gamma)\frac{\pi\,\Delta y}{L}\right)\right] - \frac{1-\gamma}{\gamma}\ln\left[\cosh\left(\wp - 2\gamma\frac{\pi\,\Delta y}{L}\right)\right]\Bigg\}, \tag{2a}$$

where we use the notation

$$S = \Gamma_1 + \Gamma_2, \qquad \gamma = \Gamma_1/S, \qquad \wp = 2\pi\big[\gamma\, y_1 + (1-\gamma)\, y_2\big]/L. \tag{2b}$$

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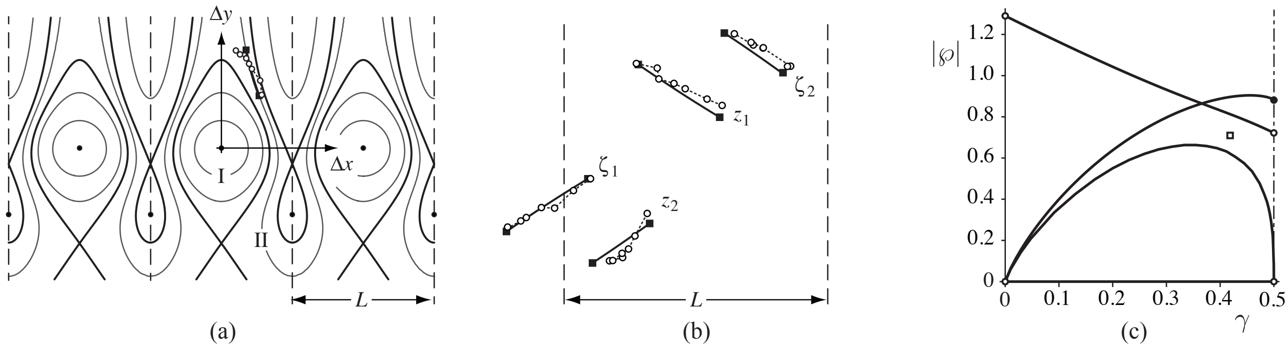

Figure 2. (a) Model phase space representation for $\gamma = 3/7$ and $\wp \approx -0.266\,\pi/L$. Filled circles mark stable fixed points, and separatrices (heavy lines) cross at unstable fixed points. The relative vortex positions from Fig. 1(a) are shown with open circles, and the corresponding model trajectory is marked by filled squares and a heavy line. (b) Real-space representation of the experimental data from Fig. 1(a) in a co-moving frame (open circles and dashed lines) and the corresponding model trajectories (filled squares and solid lines) obtained by integrating along the level curve indicated in panel (a). (c) Bifurcation diagram for phase space structure. The diagram has reflective symmetry about $\gamma = 1/2$. The line $\wp = 0$ is a bifurcation curve. Open circles indicate singular points. The open square corresponds to the example in panels (a,b).

Since the vortex strengths and the linear impulse are conserved [1], the parameters (2b) are constants of the motion. Thus, the constrained system is integrable, and the dynamics can be characterized by considering motion along level curves of $\mathbb{H}$ in the $(\Delta x, \Delta y)$ phase space. Vortex motions in real space are obtained from the phase space representation via numerical quadrature.

In Fig. 2(a) we show a representative example of the phase space for one set of parameter values. The fixed points in phase space correspond to relative equilibrium configurations of the vortices. Most initial conditions, however, lead to relative motion of the vortices. The level curves that pass through unstable fixed points, or *separatrices*, divide the phase space into distinct regimes of motion. Initial conditions from a single regime of motion give rise to qualitatively the same relative vortex motion. For example, all of the initial conditions from regime I in Fig. 2(a) give rise to motions in which the vortices with strengths Γ_1 and Γ_2 and positions z_1 and z_2, respectively, orbit one another as the entire configuration translates. Initial conditions in regime II lead to motions in which all of the vortices interact as time progresses. An example real-space trajectory from regime II is shown in Fig. 2(b) for a short duration of time; a more complete discussion of real space motions will be presented.

A comparison between the point vortex model and experimental wakes can be made by extracting the relative vortex positions, including an estimate for L, from an image such as that in Fig. 1(a). The model phase space representation is fit to the experiment, as shown in Fig. 2(a), by assuming a best-fit value of γ and then computing the value of $\wp$ based on the experimental positions. The corresponding real space comparison is obtained by fitting the background velocity U and the time scale of the motion. As shown in Fig. 2(b), this point vortex model is capable of capturing the trend of the relative vortex motion quite well, despite the many simplifications made in the analysis. Furthermore, matching the scaling of the model and the experiment provides an estimate of the experimental vorticity, which our preliminary results suggest is accurate. Thus, in addition to characterizing the net behavior of laminar four-vortex wakes, this model can provide physically relevant information about the circulations of the coherent wake vortices using only information about the relative vortex motion.

The structure of the model phase space depends on the values of the constants of the motion. Bifurcations in the phase space structure occur when separatrices connect multiple fixed points at specific values of γ and $\wp$. The bifurcation diagram, shown in Fig. 2(c), shows five different types of phase space structure. Some regimes of motion, such as regime I in Fig. 2(a), are always present, while others, such as regime II, depend on the phase space structure. We have identified a total of six unique regimes of motion. We will give examples from each of these regimes, and we will discuss those types of motion that have been observed in experiments reported in the literature and conducted by collaborators.

References

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