LAGRANGIAN AND EULERIAN HYBRID METHOD FOR WEAKLY NONLINEAR STABILITY OF A ROTATING FLOW IN A CYLINDER OF ELLIPTIC CROSS-SECTION

Yasuhide Fukumoto*a) & Youichi Mie**

*Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
**Innovation Training Program Center for R&D and Business Leaders of Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8581, Japan

Summary A hybrid method of combining the Eulerian and the Lagrangian approaches is developed for a weakly nonlinear stability analysis of a rotating flow confined in a cylinder of elliptic cross-section. The linear instability, parametric resonance between a pair of Kelvin waves, is known as the Moore-Saffman-Tsai-Widnall (MSTW) instability. When proceeding to nonlinear stage, the mean flow induced by nonlinear interaction of a Kelvin wave remains undetermined within the Eulerian framework. A steady Euler flow of an incompressible fluid is characterized as an extremal of the total kinetic energy with respect to isovortical disturbances. We develop a Lagrangian method for calculating the wave-induced mean flow by exploiting this criticality, and construct the amplitude equations to third order. The nonlinear effect saturates the MSTW modes. We then turn to three-wave interactions and have identified a resonant triad that causes the secondary instability of the stationary mode.

INTRODUCTION

Three-dimensional instability of a strained vortex tube, called the Moore-Saffman-Tsai-Widnall (MSTW) instability [1, 2], is one of ubiquitous hydrodynamic instabilities and has attracted much attention in connection with the aircraft wake turbulence over 40 years. A circular cylindrical vortex tube supports the Kelvin waves, a family of neutrally stable oscillations. The MSTW instability is caused by parametric resonance between two Kelvin waves, whose azimuthal wavenumbers m are separated by 2, via the straining field which breaks the circular symmetry. According to Krein's theory of Hamiltonian spectra, the instability is brought about by a collision of eigenvalues of positive and negative-energy waves [3]. In the presence of a basic flow, the traditional Eulerian approach encounters a difficulty in calculating wave energy, because disturbance field of second order is requested.

A steady incompressible Euler flow is characterized as an extremal of the total kinetic energy with respect to isovortical perturbations. An isovortical perturbation is directly expressible in terms of the Lagrangian variables. We develop a Lagrangian approach whereby wave energy of quadratic in amplitude is calculable solely from the linear Lagrangian displacement [4]. As a byproduct, we gain the mean flow, of second order, induced by nonlinear wave interactions [5].

With this mean flow, we are able to calculate all the coefficients of amplitude equations to third order [6, 7]. This remedies an incomplete Eulerian treatment [8]. The nonlinear evolution of the instability mode turns out to saturate the amplitude saturation. We then explore a triad resonance of Kelvin waves to account for the secondary instability.

PROBLEM SETTING OF AND THE MOORE-SAFFMAN-TSAI-WIDNALL INSTABILITY

Consider a rotating flow confined in a cylinder of infinite length, uniform along the z-axis, with elliptic cross-section whose eccentricity is measured by ε, a small parsameter. In conjunction with this distortion, the basic steady flow is represented, in terms of cylindrical coordinates (r, θ, z), as $\boldsymbol{U} = \boldsymbol{U}_0 + \varepsilon \boldsymbol{U}_1 + \cdots;\quad \boldsymbol{U}_0 = r\boldsymbol{e}_\theta,\ \boldsymbol{U}_1 = -r\sin 2\theta \boldsymbol{e}_r - r\cos 2\theta \boldsymbol{e}_\theta$, where $\boldsymbol{e}_i$ is the unit vector in the i-th direction. The term $\boldsymbol{U}_1$ of $O(\varepsilon)$ represents a pure shear.

We superimpose three-dimensional disturbance field $\tilde{\boldsymbol{u}}$ to it. We make asymptotic expansions of the velocity field in two small parameters, ε and the amplitude α, as $\tilde{\boldsymbol{u}} = \alpha \boldsymbol{u}_{01} + \epsilon\alpha \boldsymbol{u}_{11} + \alpha^2 \boldsymbol{u}_{02} + \alpha^3 \boldsymbol{u}_{03} + \cdots$. The term of $O(\alpha)$ represents the Kelvin wave and we send a pair of waves with azimuthal wavenumbers m, $m+2 (\in \mathbb{Z})$:

$$\boldsymbol{u}_{01} = A_+ \boldsymbol{u}_{A_+}(r)\mathrm{e}^{i(m\theta + k_0 z)} + B_+ \boldsymbol{u}_{B_+}(r)\mathrm{e}^{i[(m+2)\theta + k_0 z]} + A_- \boldsymbol{u}_{A_-}(r)\mathrm{e}^{i(m\theta - k_0 z)} + B_- \boldsymbol{u}_{B_-}(r)\mathrm{e}^{i[(m+2)\theta - k_0 z]}, \tag{1}$$

supplemented by complex conjugate. Here k_0 is the axial wavenumber and $A_\pm(t)$, $B_\pm(t)$ are functions of time t.

This degeneracy corresponds to intersection points (k_0, ω_0), with ω_0 being the frequency, of the dispersion curves of the m and $m+2$ waves. An analysis of $O(\varepsilon\alpha)$ shows that the degeneracy necessarily brings in instability [3, 6].

MEAN FLOW INDUCED BY NONLINEAR INTERACTION OF KELVIN WAVES

Due to degeneracy of the linear operator associated with the Euler equations, the drift current of $O(\alpha^2)$ induced by nonlinear interaction of Kelvin waves cannot be determined. Use of the Lagrangian variables overcomes this difficulty [5]. Suppose that a fluid particle located at $\boldsymbol{x}$ in the basic flow is displaced as $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{\xi}$ with $\boldsymbol{\xi}(\boldsymbol{x}, t)$ expressed in a power series as $\alpha\boldsymbol{\xi}_1 + \alpha^2\boldsymbol{\xi}_2/2 + O(\alpha^3)$. Frozen into this displacement, the vorticity is disturbed to

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \alpha\boldsymbol{\omega}_1 + \frac{\alpha^2}{2}\boldsymbol{\omega}_2;\quad \boldsymbol{\omega}_1 = \nabla \times [\boldsymbol{\xi}_1 \times \boldsymbol{\omega}_0],\ \ \boldsymbol{\omega}_2 = \nabla \times [\boldsymbol{\xi}_1 \times \boldsymbol{\omega}_1 + \boldsymbol{\xi}_2 \times \boldsymbol{\omega}_0]. \tag{2}$$

a) Corresponding author. E-mail: yasuhide@imi.kyushu-u.ac.jp

The disturbance velocity is obtained in the form of uncurl of (2).
Taking average of the disturbance velocity produces the mean flow of $O(\alpha^2)$ as

$$\overline{\boldsymbol{u}_{02}} = \frac{1}{2}\overline{\mathcal{P}\left[\boldsymbol{\xi}_1 \times \left(\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega}_0)\right)\right]}. \tag{3}$$

It is noteworthy that this second-order quantity is expressed in terms of only the first-order field $\boldsymbol{\xi}_1$.

AMPLITUDE EQUATIONS

We are ready to proceed to $O(\alpha^3)$. We confine ourselves to a non-stationary resonance of $(m, m+2) = (0, 2)$ pair. At $O(\alpha^3)$, modes of $\mathrm{e}^{ik_0 z}$ and $\mathrm{e}^{i(2\theta+k_0 z)}$ again show up, and imposition of the solvability condition yields the amplitude equations. In order to make the resulting equations Hamiltonian, we introduce normalized amplitudes $z_{1\pm} = A_\pm \mathrm{e}^{i\omega_0 t}/\sqrt{|p|}$, $z_{2\pm} = \overline{B_\pm}\mathrm{e}^{-i\omega_0 t}/\sqrt{|q|}$. Then we are led to

$$\begin{aligned}
\frac{\mathrm{d}z_{1\pm}}{\mathrm{d}t} &= i\left[\epsilon\left(\sigma\overline{z_{2\pm}} - p_{12}z_{1\pm}\right) + \alpha^2 c_{15} z_{1\mp}\overline{z_{2\pm}}z_{2\mp} + \alpha^2 z_{1\pm}\left(c_{11}|z_{1\pm}|^2 + c_{12}|z_{2\pm}|^2 + c_{13}|z_{1\mp}|^2 + c_{14}|z_{2\mp}|^2\right)\right], \\
\frac{\mathrm{d}z_{2\pm}}{\mathrm{d}t} &= i\left[\epsilon\left(\sigma\overline{z_{1\pm}} - p_{22}z_{2\pm}\right) + \alpha^2 c_{25} z_{2\mp}\overline{z_{1\pm}}z_{1\mp} + \alpha^2 z_{2\pm}\left(c_{21}|z_{1\pm}|^2 + c_{22}|z_{2\pm}|^2 + c_{23}|z_{1\mp}|^2 + c_{24}|z_{2\mp}|^2\right)\right],
\end{aligned} \tag{4}$$

where σ is the growth rate. If the coefficients satisfy the relation $c_{12} = c_{21},\ c_{14} = c_{23},\ c_{15} = c_{25}$, (4) becomes canonical Hamiltonian equations. We confirm numerically that this is indeed the case.
The freedom of the Hamiltonian system (4) is four, while only three first integrals are available, namely, the Hamiltonian, the disturbance energy to $O(\alpha^2)$, and the flow flux in the z-direction $J = k_0\left(|z_{1+}|^2 - |z_{2+}|^2 - |z_{1-}|^2 + |z_{2-}|^2\right)$. In this keeping, the solution of (4) exhibits chaotic behavior, but amplitude growth is suppressed.

SECONDARY INSTABILITY BY THREE-WAVE INTERACTION

The secondary instability of the MSTW mode is possibly attributable to three-wave resonance [9]. We search for a resonance triad including the stationary resonance between helical modes ($m = \pm 1$). A candidate is found from interaction of waves of $m = \pm 1$, 3 and 4. The second radial mode of $m = \pm 1$ is located at $(k_0, \omega_0) \approx (\beta, 0)$ with $\beta \approx 3.286$. We observe from the dispersion curves that a resonance triad with this mode is constituted by $(m, k_0, \omega_0) \approx (3, \beta/2, 3.32)$ and $(-4, -3\beta/2, -3.32)$ (Figure 1) [10].

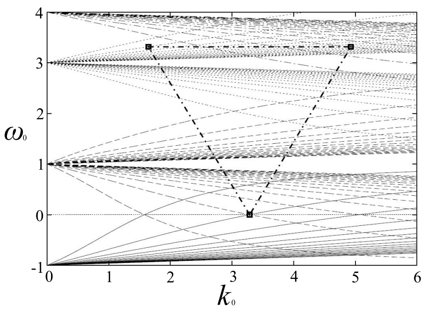

Figure 1. Dispersion relations of Kelvin waves, of $m = \pm 1, 3, 4$

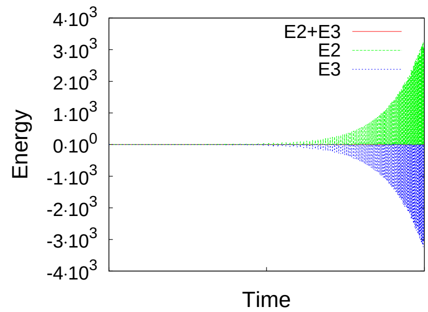

Figure 2. Evolution of energy of $m = 3, 4$ waves

The amplitude equations of these three Kelvin waves are derived without difficulty. By a careful adjustment of the initial condition in such a way that the total energy is close to zero, the amplification of wave energies of $m = 3$ and $m = 4$ are realized (Figure 2). Triad resonance is a plausible mechanism to excite a number of waves once the MSTW mode grows.

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