THEORETICAL RELATIONS BETWEEN QUANTITIES AT A WATER SURFACE AND THE INSTANTANEOUS SUBSURFACE FLOWFIELD

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1. INTRODUCTION. This communication considers how to estimate subsurface flow from the fluctuating velocities and height at the surface of a river or sea, supposed to be accessible from high-resolution, high-speed video recordings, perhaps by a stereo pair of bank-mounted cameras. A *kinematic* relation will be derived in section 2, while the Navier-Stokes equations lead to *dynamical* relations between quantities at the surface with the flow field below, as discussed for constant-density fluid in section 3.

Empirical relations might also be used to estimate subsurface flow. For example, Large Eddy Simulation (LES) permits statistical correlations between the surface and subsurface flow to be estimated. Some relevant results by our laboratory, based on the Proper Orthogonal Decomposition, have been presented in Nguyen *et. al* (2010). A weakness of such empirical estimation methods is that if the actual flow includes events are not "spanned" by samples in the LES database, the predictions will fail. Correspondingly it is important to have other tools available, such as the kinematic and dynamical relations derived here. Numerical tests of these relations will be provided at the Congress.

2. KINEMATICAL RELATIONS. In the current scenario, we want to estimate the subsurface vorticity field $\boldsymbol{\omega}(\mathbf{x})$, defined as the curl of the velocity field $\mathbf{u}(\mathbf{x})$, from the velocity field at the free surface.

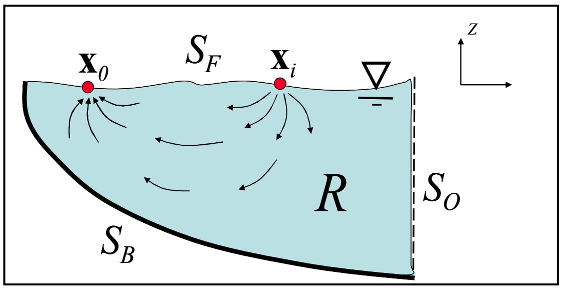

Fig. 1. Cross section through the volume R over which integrations are performed. *R* is bounded by a closed surface *S,* the union of three surfaces S_B, S_O, and S_F. Only the free surface S_F may evolves in time. The river bank (or seabed) S_B is roughly perpendicular to the page, while the "open boundary" S_O is taken to be prismatic with a vertical generator. A typical incompressible "analyzing flow" $\nabla\phi$ ($\nabla\psi$ in section 3), set up by a point source at $\boldsymbol{x}_i$ and a point sink at $\boldsymbol{x}_0$, is sketched by some curved arrows.

Consider the flow region *R,* sketched in Fig. 1, enclosed by a surface *S* comprising the free surface S_F, a no-slip river bottom S_B, and a roughly vertical "open" boundary S_O through which the water flows in and out. Only S_F is considered to vary in time. In *R*, we evaluate the integral $\int_R \boldsymbol{\omega}\times\nabla\phi\, dV$, where the "analyzing flow" $\nabla\phi$ is assumed incompressible, possibly excepting some isolated point sources or dipoles within *R* or on *S*. Fig. 1 shows a typical choice of $\nabla\phi$, chosen to sample vorticity near $\boldsymbol{x}_i$ as explained shortly. This volume integral can be transformed into a surface integral by substituting $\omega = \nabla\times\mathbf{u}$, integrating by parts, invoking $\nabla\cdot\mathbf{u}=0$, and applying the divergence theorem. The result is

$$\int_R \boldsymbol{\omega}\times\nabla\phi dV = \int_S (\mathbf{u}\nabla\phi - \phi\nabla\mathbf{u})\cdot\mathbf{n}dS \tag{1}$$

The right-hand side is thus a weighted integral over *S* of the velocity, and gradients of its normal component. The weight depends on the solution of the Laplace equation for ϕ, which in turn depends on the boundary conditions for ϕ on *S*. At the riverbed, we invoke the no-slip condition $\mathbf{u}=0$ which kills both RHS contributions from S_B because $\nabla\mathbf{u}\cdot\mathbf{n}=0$ there (though this "pushes" all the boundary layer vorticity inside *R*, where it must be estimated by our procedure). Since little or no information will be available at the open boundary S_O, it is clearly attractive to specify $\nabla\phi\cdot\mathbf{n}=0$ and thereby eliminate contributions from the first RHS term there. In the example of $\nabla\phi$ in Fig. 1, we specify a point source at $\boldsymbol{x}_i$ on S_F, and a point sink at a surface point $\boldsymbol{x}_0$; matching the strength of the sink to that of the source, and specifying $\nabla\phi\cdot\mathrm{n}=0$ on all boundary points except for $\boldsymbol{x}_0$ and $\boldsymbol{x}_i$, assures no net outflow through S_O. This specification for ϕ will then extract the contributions to the integrand, $\boldsymbol{\omega}\times\nabla\phi$, from regions around $\boldsymbol{x}_i$ and $\boldsymbol{x}_0$. By "scanning" $\boldsymbol{x}_i$ across the free surface S_F in both streamwise and cross-stream directions, with $\boldsymbol{x}_0$ fixed, and evaluating the equation (1) for each $\boldsymbol{x}_i$, I hope to get information on the distribution of vorticity near the free surface. To evaluate the integrals, we could divide R into J discrete volumes R_j whose sizes are small compared to the scale of variation of ϕ , and broaden the "point" source and sink to cover finite patches S_0 and S_i on the boundary where i ranges from 1 to I. Similarly S_O would be discretized into K patches S_k . Then for each patch S_i , the LHS of (1) yields a weighted sum of the unknown values of vorticity at each R_j, while the RHS would yield a known value from measured values at the water surface of $\mathbf{u}$ and $-\nabla_{2D}\cdot\mathbf{u}$ (the latter being equal to the longitudinal gradient of normal velocity $\frac{\partial u_n}{\partial n}$), plus a sum of unknown contributions from the subsurface patches S_k. This would yield a system of I linear equations, with $J+K$ unknowns. Normally $J+K>I$, *i.e.* the system would be underdetermined. However Singular Value Decomposition (SVD) can still obtain a solution that minimizes the norm of the solution, and this could provide a "best estimate" of the subsurface field of vorticity and of the unknown velocity gradients on S_O.

3. DYNAMICAL RELATIONS. The divergence of the “Lamb form” of the Navier-Stokes equation yields the following Poisson equation;

$$\nabla^2(gE)=\nabla\cdot(\mathbf{u}\times\boldsymbol{\omega}) \tag{2}$$

in which $E=\frac{u^2}{2g}+\frac{p}{\rho g}+z$ is the total head. Accordingly, a subsurface distribution of $\mathbf{u}\times\boldsymbol{\omega}$ leaves an imprint on E at the free surface. If wind stresses are negligible, then pressure at the surface is constant and E is then the sum of surface height and velocity head. By the same procedure as the last section, we can derive a solution to the Poisson equation (4) at surface points. Following Chang (1992) we take the dot product of the Lamb form of the Navier-Stokes equation with an irrotational vector field $\nabla\psi$, and again integrate over the domain R sketched in Fig. 1. Integrating by parts and invoking div $\mathbf{u}$ = 0, one obtains

$$\frac{1}{\rho}\int_S E\,\nabla\psi\cdot\mathbf{n}dS=\int_R(\mathbf{u}\times\boldsymbol{\omega})\cdot\nabla\psi dV+\int_S\psi\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{n}dS \tag{3}$$

in which the viscous boundary term has been neglected. If we choose the same boundary conditions on ψ as for ϕ in section 2, this will extract E at $\boldsymbol{x}_i$ on the water surface. (If there were dynamic pressure data from the bed, one could additionally take one $\boldsymbol{x}_i$ at the position of the pressure gage(s).) This choice yields

$$\frac{1}{\rho}(E(\mathbf{x}_i)-E(\mathbf{x}_0))=\int_R(\mathbf{u}\times\boldsymbol{\omega})\cdot\nabla\psi dV+\int_S\psi\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{n}dS \tag{4}$$

To evaluate the surface integrals will require an accurate determination of the fluctuating water level and its temporal **rate of change.** Additionally, there will be a contribution from S_O. Analogous to the last section, one could take the integrand $\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{n}$ there to be unknown. Data from I surface points $\boldsymbol{x}_i$ would again allow one to set up a system of I linear equations, whence estimates for the unknown field of $\mathbf{u}\times\boldsymbol{\omega}$ could be computed.

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