TACKLING STRUCTURAL COMPLEXITY BY JONES' POLYNOMIALS

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Summary In this paper we present new results based on applications of knot theory to tackle and quantify structural complexity of vortex dynamics. In the ideal case of Euler equations we show that the topology of any vortex tangle, made by knots and links, can be identified and described by the Jones's polynomials of the tangle, expressed in terms of kinetic helicity. Explicit calculations of the Jones polynomial for the left-handed and right-handed trefoil knots and for the Whitehead link via the figure-of-eight knot are worked out for illustration. This novel approach contributes to establish a fundamentally new paradigm in topological fluid mechanics, by extending the former interpretation of helicity in terms of linking numbers to the much richer context of knot polynomials, and by opening up new directions of work both in mathematical fluid dynamics and in numerical diagnostics of vortex flows.

INTRODUCTION

Helicity plays a fundamental role in fluid mechanics, being it an invariant of the Euler equations and a robust quantity of the dissipative Navier–Stokes equations. Kinetic helicity is defined by

$$H \equiv \int_\Omega \mathbf{u} \cdot \boldsymbol{\omega}\, d^3\mathbf{x}\,, \tag{1}$$

where $\mathbf{u}$ is the velocity field, defined on an unbounded domain of $\mathbb{R}^3$, and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity, defined on a sub-domain Ω. For simplicity we assume $\nabla \cdot \mathbf{u} = 0$ everywhere, and we request $\boldsymbol{\omega} \cdot \hat{\mathbf{n}} = 0$ on $\partial\Omega$, where $\hat{\mathbf{n}}$ is orthogonal to $\partial\Omega$, with $\nabla \cdot \boldsymbol{\omega} = 0$.

Let $\vec{\mathcal{K}}$ be an *oriented* tangle of knots given by a collection of N thin vortex filaments in $\mathbb{R}^3$. The topological interpretation of helicity in terms of linking numbers was established originally by Moffatt (1969) and extended by Moffatt & Ricca (1992) as

$$H(\vec{\mathcal{K}}) = \sum_i \kappa_i^2 Lk_i + 2\sum_{ij} \kappa_i \kappa_j Lk_{ij}\ , \tag{2}$$

where κ_i denotes the ith vortex circulation, Lk_i the (Călugăreanu-White) self-linking number of the ith vortex $\vec{\gamma}_i$, and Lk_{ij} the (Gauss) linking number of $\vec{\gamma}_i$ with $\vec{\gamma}_j$. As the examples of the Whitehead link and the Borromean rings demonstrate, the linking number sometimes fails to detect essential topology. A richer topological description is therefore needed to extend the topological characterization provided by helicity and linking numbers.

Since helicity is an abelian Chern-Simons (CS) action of classical field theory, Jones polynomials seem the most appropriate invariants to classify the topology of vortex knots and links under the Euler equations. Moreover, given the extraordinary success of topological quantum field theory (TQFT), that provides a field theoretical justification for knot invariants beyond quantum groups (Witten, 1989), it is natural to expect that an approach similar to that used in CS-TQFT can be extended to the study of complex fluid flows via helicity.

JONES' POLYNOMIAL OF VORTEX KNOTS AND LINKS VIA HELICITY

By reducing eq. (1) to a loop integral (Barenghi *et al.*, 2001), that is

$$H = \sum_i \kappa_i \oint_{\vec{\gamma}_i} \mathbf{u} \cdot d\mathbf{l}\,, \tag{3}$$

where $\mathbf{u}$ is the vortex self-induced velocity of the Biot-Savart law, we can show that t^H satisfies the skein relations of knot polynomials. We have:

Theorem (Liu & Ricca, 2012): *Let $\vec{\mathcal{K}}$ denote a vortex knot (or an n-component link) of helicity $H = H(\vec{\mathcal{K}})$. Then*

$$t^{H(\vec{\mathcal{K}})} = t^{\oint_{\vec{\mathcal{K}}} \mathbf{u} \cdot d\mathbf{l}} \tag{4}$$

satisfies the skein relations of the Jones polynomial $V = V(\vec{\mathcal{K}})$.

Detailed demonstration of this is provided by Liu & Ricca (2012), where proof is based on the derivation of the Kauffman bracket polynomial for *unoriented* knots, then extended to oriented knots. Knot polynomials are (standardly) constructed by analyzing the crossing states in a knot diagram. This resorts to the analysis of the elementary states given by the over-crossing $L_+ =$, the under-crossing $L_- =$ and the disjoint union with a trivial circle $\mid \sqcup \bigcirc$, where $\mid$ denotes the rest of a link component. A technique called "local path-addition" leads then to the computation of the skein ralations of the Jones' polynomial in terms of $t^{H(\vec{\mathcal{K}})}$.

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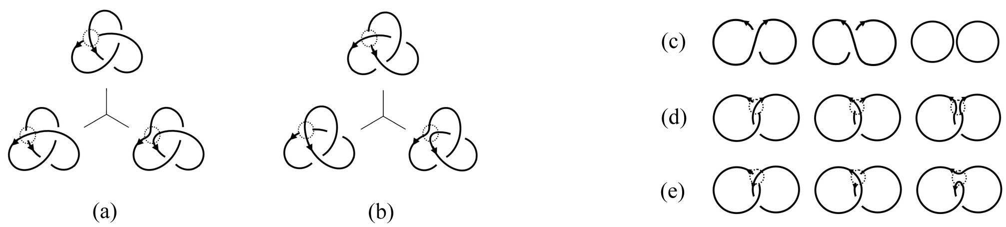

Figure 1. (a) Left-handed and (b) right-handed trefoil knot decomposed by standard reduction techniques. The corresponding Jones' polynomials are obtained by calculations of the elementary states given by the (c)-(e) diagrams.

LEFT- AND RIGHT-HANDED TREFOIL KNOTS AND WHITEHEAD LINK

For the sake of illustration we work out the Jones' polynomials by considering the following examples. First let us consider the left-handed trefoil knot $\vec{T}^L$ and the right-handed trefoil knot $\vec{T}^R$, shown in Figure 1(a) and 1(b).

The computation of the Jones' polynomial is based on application of reduction techniques performed recursively on crossing sites. Based on the polynomials of the disjoint union of trivial circles (see the diagram of Figure 1c) and of the Hopf links (diagrams 1d and 1e), we obtain the Jones' polynomials for the left-handed trefoil knot $V(\vec{T}^L) = \tau^{-1} + \tau^{-3} - \tau^{-4}$, and for the right-handed trefoil knot $V(\vec{T}^R) = \tau + \tau^3 - \tau^4$.

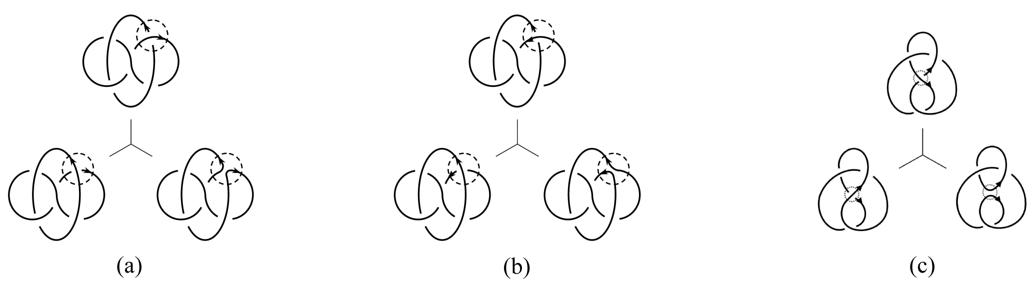

Figure 2. Diagrams (a) and (b) show Whitehead links of different orientation. The figure-of-eight knot is shown in diagram (c).

A second example is provided by the Whitehead link W (see Figure 2). By applying standard reduction techniques, we are led to the computation of the Jones' polynomial for the figure-of-eight knot $\vec{F}^8$ (see Figure 2c), $V(\vec{F}^8) = \tau^{-2} - \tau^{-1} + 1 - \tau + \tau^2$, and of the Hopf links and the trefoil knots as above. Hence, the Jones' polynomial of the Whitehead link is given by $V(\vec{W}_+) = \tau^{-\frac{7}{2}} - 2\tau^{-\frac{5}{2}} + \tau^{-\frac{3}{2}} - 2\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} - \tau^{\frac{3}{2}}$, irrespective of the orientation of its components.

NEW STRUCTURAL COMPLEXITY MEASURES FOR COMPLEX FLOWS

Knot polynomial invariants provide a powerful tool to investigate complex topologies of vortex flows. Numerical implementation of knot diagram analysis of complex tangles of filaments is straightforward (see, for instance, Barenghi *et al.*, 2001) and reduction techniques to compute Jones' polynomials of knots and links can be implemented as well for any large number of components. In presence of dissipation, then, vortex reconnection takes place and there is a continuous change of vortex topology. Jones's polynomial are no longer invariant quantities, but, as helicity, their continuous change in time provides a time-dependent information on the real-time structural evolution of vorticity. Information obtained by the numerical implementation of these new concepts can therefore be profitably used in visiometrics and numerical diagnostics to investigate aspects of structural complexity and energy-complexity relations in turbulent flows (Ricca, 2009).

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