STABILITY, RESONANCES AND INSTABILITY OF THE STEADY ROTATION OF A SYSTEM OF THREE EQUIDISTANT VORTICES OUTSIDE A CIRCLE

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Summary A complete non-linear analysis of the stability of the steady rotation of three point vortices, placed in a plane at the vertices of a regular triangle outside a circular domain is carried out using the results of the Kolmogorov-Arnold-Moser theory. All the resonances of up to fourth order inclusive encountered here are listed and studied. The investigations of Havelock who solved this problem in a linear formulation are thereby completed.

INTRODUCTION

A system of point vortices has been studied from different points of view in numerous papers (see the reviews [1–4]).The stability of the steady rotation of a system of seven identical point vortices, located in a plane at the vertices of a regular heptagon, has been proved [5, 6].

Together with the known results of many authors, this led to the complete solution of the Kelvin problem of 1878, that is, to the conclusion that the maximum number of vertices of a stable regular n-sided polygon with a point vortex at each of the vertices is equal to seven.

The Kelvin problem, extended to the case of a sphere and a circle (see [7–10] together with the references available there), has also been studied in an exact non-linear formulation.

The problem of the stability of a system of n point vortices, uniformly arranged on a circle of radius R_0 outside a circle of radius R with a common centre of symmetry (see Fig. 1), has been solved [11] in a linear formulation. It was established that the corresponding linearized system has an exponentially increasing solution when $n > 7$ and, also, if $2 \le n \le 6$ and the parameter $q = R^2/R_0^2$ is greater than a certain critical magnitude $q_* = q_*(n)$: $q_* < q < 1$. A critical case occurs in all the remaining cases in the stability problem: the eigenvalues of the linearization matrix have null real parts.

The problem is studied below in the exact non-linear formulation for $n = 3$ when $0 < q \le q_*$. The necessary and sufficient conditions for stability are obtained. The problem is broken down into several problems that require an individual approach and, in particular, the use of methods of the Kolmogorov-Arnold-Moser theory. All the resonances encountered here up to the fourth order inclusive are analysed below using the results of Markeyev and Sokol'skii [12, 13].

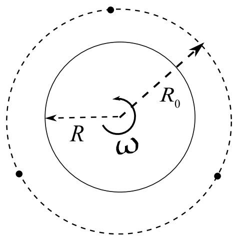

Fig. 1

FORMULATION OF THE PROBLEM. FORMULATION OF THE RESULTS

The motion of a system of n point vortices in a plane outside a circle of radius R, assuming the flow around the circle is irrotational, is described by the equations [11, 14]

$$\dot{z}_k^* = \frac{1}{2\pi i}\left(\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\varkappa_j}{z_k - z_j} - \sum_{j=1}^{n} \frac{\varkappa_j}{z_k - \widehat{z}_j} + \frac{1}{z_k}\sum_{j=1}^{n}\varkappa_j\right), \quad k = 1, \dots, n. \tag{1}$$

Here, $z_k = x_k + iy_k$ $(k = 1, \dots, n)$ are complex variables, x_k and y_k are the Cartesian coordinates of the k-th vortex, $\varkappa_k$ is its intensity and $\widehat{z}_k = R^2/z_k^*$ is the reflection of the k-th vortex by the boundary of the circle.

System (1) is a Hamiltonian system with a Hamiltonian

$$H = -\frac{1}{4\pi}\sum_{1\leqslant j<k\leqslant n} \varkappa_j \varkappa_k \ln\left|z_j - z_k\right|^2 + \frac{1}{8\pi}\sum_{j=1}^{n}\sum_{k=1}^{n} \varkappa_j\varkappa_k\left[\ln\left|R^2 - z_j z_k^*\right|^2 - 2\ln\left|z_k\right|^2\right]. \tag{2}$$

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We will henceforth assume that all the vortices have an identical intensity $\varkappa$. System (1) has the exact solution

$$z_k(t) = e^{i\omega t} u_k, \quad u_k = R_0 e^{2\pi i(k-1)/n}; \quad k = 1, 2, \dots, n, \qquad (3)$$
$$\omega = \frac{\varkappa}{4\pi R_0^2}\left(3n - 1 - \frac{2n}{1-q^n}\right), \quad q = \frac{R^2}{R_0^2} < 1.$$

Hence, a system of identical vortices arranged on a circle of radius R_0 at the vertices of an inscribed regular n-sided polygon (of the "vortex" n-sided polygon) rotates at a constant angular velocity $\omega = \omega(q)$ (Fig. 1).

Note that the conditions for steady rotation (3) are clearly unstable for any n in the Lyapunov sense. Actually, if a regular vortex n-sided polygon is perturbed at the initial instant $t = 0$, such that it remains regular but of another size, then the subsequent motion will be a uniform rotation as before, but with a different angular velocity. As a result, no matter how small the perturbation is initially, with time it becomes of the order of the diameter of the polygon. The linearly increasing solution of the linearized system and the Jordan block of the 2×2 linearization matrix, corresponding to its twofold zero eigenvalue, conforms with this obvious instability. This situation is typical of the steady motions of Hamiltonian systems when there are cyclic coordinates.

Since Lyapunov stability is impossible in the case of a permanent rotation, we will refine the definitions. By a change of variables, we will change from system (1) to a system with a cyclic coordinate. In the theory of the steady motion of such systems developed by Routh [15], account is taken of the fact that a steady motion is always unstable with respect to a cyclic coordinate (the perturbations increase linearly with time).The discussion must therefore be about stability with respect to part of the variables, the positional coordinates and the momenta. Eliminating the momentum corresponding to the cyclic coordinate from the system, we arrive at a reduced Hamiltonian system. By Routh stability of the steady rotation (3), we shall mean the Lyapunov stability of the equilibrium position of this reduced system.

Correspondingly, instability of a vortex polygon will imply the Lyapunov instability of this equilibrium position. The result of the paper is a criterion for the stability of the steady rotation (3) of the vortex triangle, shown schematically in Fig. 2. The domains and points of stability (instability) are labelled with a plus (minus) sign.

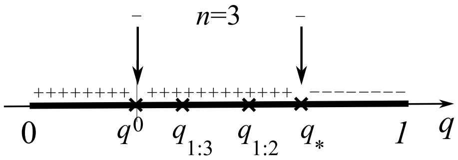

Fig. 2

The resonances [12, 13] in which: q^0 is a twofold zero (the diagonalizable case), $q_{k:m}$ is a $k : m$ resonance and q_* is a 1:1 resonance (the non-diagonalizable case) correspond to the critical values of the parameter q

$$q^0 \approx 0.2625, \quad q_{1:3} \approx 0.2709, \quad q_{1:2} \approx 0.2724, \quad q_* \approx 0.2737 \qquad (4)$$

In conclusion, we note that a complete non-linear analysis of the problem considered here in the case of n vortices ($n = 2, 4, 6$) has been presented in [16].

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