THE CIRCULATION MODEL OF VORTEX FLOW OF A VISCID WALL LAYER

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Summary The problem of unsteady circular layered flow of viscous fluid in thin flat channel with curvilinear lateral boundaries is considered. The top and the bottom boundaries have given velocity distribution. This problem requires the solution of hydrodynamical problem for unsteady circular flow on the top surface of the layer. We apply well known discrete vortex method for numerical solution of the problem, for which complex potential and velocity field is written in a discrete form. This model allows to solve different problems for different values of viscosity (high viscosity case was discovered by Couette M. (1890) and Hele-Shaw H.S (1898)).

The report is devoted to an application of vortex methods in the modeling of the flow for the shallow channel with complex boundaries. The 3D layer flow of viscous fluid between two flat surfaces is described by Navier-Stokes equation

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\nabla)\vec{V} = -\frac{\nabla p}{\rho} + \vec{F} + \frac{\mu}{\rho}\Delta\vec{V} \tag{1}$$

$$\nabla\vec{V} = 0\,, \tag{2}$$

with boundary conditions for moved boundaries

$$\vec{V}(x,y,0,t) = \vec{V}_0(x,y,t)\,,\ \vec{V}(x,y,h,t) = \vec{V}_h(x,y,t) \tag{3}$$

In an assumption that the flow is layered, we found the solution in following form

$$\vec{V}(x,y,z,t) = (u(x,y,z,t), v(x,y,z,t), 0)\,, \tag{4}$$

$$p = p(x,y,z,t)\,,\ \ \rho = const$$

Then we suppose that differential function $\varphi = \varphi(x,y,z,t)$ is determined for various z and t. This function describes the 3D distribution of 2D velocity ($u(x,y,z,t)$, $v(x,y,z,t)$) using following equation

$$u(x,y,z,t) = \frac{\partial}{\partial x}\varphi(x,y,z,t)\ ,\quad v(x,y,z,t) = \frac{\partial}{\partial y}\varphi(x,y,z,t)\cdot \tag{5}$$

Using (4,5), the solution of eq.(1) can be written in the integral form

$$\frac{\partial}{\partial t}\varphi(x,y,z,t) + \frac{1}{2}\left(\left(\frac{\partial}{\partial x}\varphi(x,y,z,t)\right)^2 + \left(\frac{\partial}{\partial y}\varphi(x,y,z,t)\right)^2\right) + \frac{P(x,y,t)}{\rho} - \frac{\mu}{\rho}\frac{\partial^2}{\partial z^2}\varphi(x,y,z,t) = q(z,t) \tag{6}$$

Equation (6) is nonlinear equation, but it is linear equation in respect to the second z-derivative. To achieve correct solutions for steady and unsteady regimes with given viscosity and velocity of boundaries it is necessary to carry out an asymptotic transformation of the solution (6), not the equation (1).

For steady laminar (Re<10^3 due to the depth of the channel) flows the solution (6) can be presented in the following form

$$P(x,y) - \mu\frac{\partial^2}{\partial z^2}\varphi(x,y,z) = \rho q(z)\cdot \tag{7}$$

which has the solution

$$\varphi(x,y,z) = \frac{z^2}{2\mu}P(x,y) + z\varphi_1(x,y) + \varphi_0(x,y) - \frac{\rho}{\mu}Q(z)\,, \tag{8}$$

Depending on the values of velocity of boundaries (3), the eq.(8) determines different solutions, e.g.: Poiseuille flow [5], Couette flow [2] or Hele-Shaw flow [4].

For unsteady slow laminar flow the solution (6) is linear equation as following

$$\frac{\partial}{\partial t}\varphi(x,y,z,t) + \frac{P(x,y,t)}{\rho} - \frac{\mu}{\rho}\frac{\partial^2}{\partial z^2}\varphi(x,y,z,t) = q(z,t) \tag{9}$$

To find the solution of this equation we have to add unsteady boundary condition, which can be found from another hydrodynamical problem, described below.

Consider the mathematical problem for potential $\varphi = \varphi(x,y,h,t)$ in the multi-connected region D^+ with both impermeable fixed boundary L_d and free moving boundary L_V. Substitution of equation (5) in equation (2) gives Laplace equation for potential function [1,3,8]:

$$\Delta\varphi = 0\ ,\quad \text{at}\quad D^+ \tag{10}$$

with boundary conditions (fig.1)

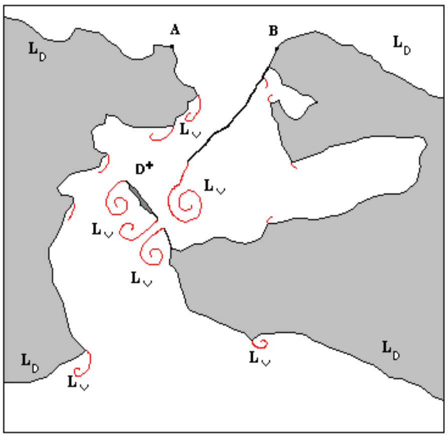

Fig.9. Geometry of the channel

$$\int_A^B (\nabla\varphi, n)ds = Q_S \quad \text{at} \quad AB, \tag{11}$$

where Q_S is water discharge in the channel,

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{at } L_d, \qquad \left.\frac{\partial\varphi}{\partial n}\right|^+ = \left.\frac{\partial\varphi}{\partial n}\right|^- \quad \text{at} \quad L_V, \tag{12}$$

$$\frac{\partial\varphi^+}{\partial t} + \frac{\left|\nabla\varphi^+\right|^2}{2} = \frac{\partial\varphi^-}{\partial t} + \frac{\left|\nabla\varphi^-\right|^2}{2} \quad \text{at} \quad L_V \tag{13}$$

and initial condition

$$L_d,\ L_O = L_V(t_O), \qquad \varphi^+\Big|_{t=0} = \varphi_0^+ \tag{14}$$

$$\text{as well as} \quad \left|\nabla\varphi^+\right| < \infty. \tag{15}$$

We introduce the complex potential $\Phi(\xi, t)$ and velocity field in the following form [3]

$$\Phi(\xi,t) = \varphi + i\psi = \frac{1}{2\pi i}\int_{L_d} f(\omega,t)\ln(\xi-\omega)d\omega + \frac{1}{2\pi i}\int_{L_v(t)} f(\omega,t)\ln(\xi-\omega)d\omega, \tag{16}$$

$$\overline{V}(\xi,t) = \frac{\partial\Phi(\xi,t)}{\partial z} = u - iv = \frac{1}{2\pi i}\int_{L_d}\frac{f(\omega,t)}{\xi-\omega}d\omega + \frac{1}{2\pi i}\int_{L_v(t)}\frac{f(\omega,t)}{\xi-\omega}d\omega \tag{17}$$

Then we solve this problem for fixed L_d and moving L_V boundaries. For numerical solution of the (10)-(15) we apply well known discrete vortex method [3,6,7], for which complex potential (16) and velocity field (17) can be written as following

$$\Phi(\xi,t) = \varphi + i\psi = \sum_{j_d=1}^{M_d}\frac{\Gamma_{j_d}(t)}{2\pi i}\ln(\xi-\omega_{j_d}) + \sum_{j_v=1}^{N_v(t)}\frac{\Gamma_{j_v}}{2\pi i}\ln(\xi-\omega_{j_v}(t)), \tag{18}$$

$$\overline{V}(\xi,t) = u - iv = \sum_{j_d=1}^{M_d}\frac{\Gamma_{j_d}(t)}{2\pi i(\xi-\omega_{j_d})} + \sum_{j_v=1}^{N_v(t)}\frac{\Gamma_{j_v}}{2\pi i(\xi-\omega_{j_v}(t))}, \tag{19}$$

where

$$\Gamma_{j_d}(t) = \int_{L_{j_d}} f(\omega,t)d\omega, \qquad \Gamma_{j_v} = \int_{L_{j_v}(t)} f(\omega,t)d\omega \qquad j_d = \overline{1,M_d} \qquad j_v = \overline{1,N_v(t)}. \tag{14}$$

Small depth of the channel between flat boundaries is the main feature of flow under consideration. As a result, we can apply the mathematical problem with multi-connected region and complex boundaries. This solution gives averaged distribution of velocity field over the depth of the channel. Analysis shows that numerical results and experimental data coincide quite well [1].

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