INTERACTIONS OF TWO COAXIAL DYSON'S VORTEX RINGS INSIDE A CIRCULAR CYLINDER

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Summary An axisymmetrical problem on a motion of a system of vortex rings with the small, but nonzero circular cross section, inside a circular cylinder in approaching of an ideal incompressible fluid is considered. To satisfy boundary conditions a system of imaginary vortex filaments are introduced. They are placed on a fixed distance from the boundary. Researches show that influence of solid surfaces results in decreasing of axial self-induced velocity of rings. It was found that two vortex rings inside a circular cylinder can participate in a periodical motion by analogy with the periodical motion in unbounded space (first discovered by Helmholtz H. in 1858).

In modern hydromechanics the method of images is actively used at the modeling different vortex flows. The theoretical ground of this method with reference to vortex flows in an ideal incompressible fluid was indicated yet by H.Helmholtz [1]. This method had found wide enough applications in XX century in the study of the flow characteristics of wings [2,3]. In spite of wide applications of method of images in point vortex dynamics, its application for axisymmetrical vortex flows appears by limited one in approaching of an ideal incompressible fluid, namely: an interaction of vortex rings with the flat wall located perpendicular to the vortex plane, a motion of vortex ring near (or inside) a spherical surface (see details [4,5]).

Consider a single axisymmetrical vortex ring of radius R, intensity Γ, small cross section radius $a=nR$ ($n<<1$) and axial position Z in a cylindrical coordinate system. Let the flow inside the vortex ring is rotational while the external flow is potential. Then the stream function distribution is described by (so called Dyson's vortex ring [6])

$$\Psi(r,z)=\frac{\Gamma}{2\pi}\sqrt{R\,r}\left[\left(\frac{2}{k}-k\right)\mathbf{K}(k)-\frac{2}{k}\mathbf{E}(k)\right], \qquad (1)$$

where $\mathbf{K}(k)$, $\mathbf{E}(k)$ are full elliptic integral of the first and the second kind accordingly, $k^2=4R\,r/[(R-r)^2+(Z-z)^2]$.

If the system of N coaxial vortex rings of intensity Γ_α, radius R_α, cross section radius $a_\alpha=n_\alpha R_\alpha$ and axial position Z_α ($\alpha=1,...,N$) are moved inside an endless circular cylinder of radius R_0, then, following the method of images, to satisfy boundary condition on the cylindrical surface we have to introduce a system of M coaxial circular vortex filaments (fig.1) of intensity Γ_β, radius $R_C>R_0$ and axial position $Z_\beta=Z_C+\beta\Delta_Z$ ($\beta=1,...,M$) [7]. To determine intensities Γ_β we introduce a system of M collocation points (R_0, Z_0+Z_β), where values of stream function induced by system of real and imaginary vortices have to be equal. As a result, we obtain the following system of algebraic equations

$$\left[\left(\frac{2}{k_{m\beta}}-k_{m\beta}\right)\mathbf{K}(k_{m\beta})-\frac{2}{k_{m\beta}}\mathbf{E}(k_{m\beta})\right]\Gamma_\beta=\sum_{\alpha=1}^{N}\sqrt{\frac{R_\alpha}{R_C}}\left[\left(\frac{2}{k_\alpha}-k_\alpha\right)\mathbf{K}(k_\alpha)-\frac{2}{k_\alpha}\mathbf{E}(k_\alpha)\right], \qquad (2)$$

where $k_{m\beta}^2=4R_0R_C/[(R_0+R_C)^2+(Z_\beta-Z_m)^2]$, $k_\alpha^2=4R_0R_\alpha/[(R_0+R_\alpha)^2+(Z_\alpha+Z_m)^2]$, $Z_m=Z_0+m\Delta_z$.

Therefore stream function for the flow inside the circular cylinder is determined by contribution of both real and imaginary vortex rings

$$\Psi(r,z)=\sum_{\alpha=1}^{N}\frac{\Gamma_\alpha}{2\pi}\sqrt{R_\alpha r}\left[\left(\frac{2}{k_\alpha^*}-k_\alpha^*\right)\mathbf{K}(k_\alpha^*)-\frac{2}{k_\alpha^*}\mathbf{E}(k_\alpha^*)\right]+\sum_{\beta=1}^{M}\frac{\Gamma_\beta}{2\pi}\sqrt{R_C\,r}\left[\left(\frac{2}{k_\beta^*}-k_\beta^*\right)\mathbf{K}(k_\beta^*)-\frac{2}{k_\beta^*}\mathbf{E}(k_\beta^*)\right], \qquad (3)$$

where $[k_\alpha^*]^2=4R_\alpha r/[(R_\alpha-r)^2+(Z_\alpha-z)^2]$. Fig.2 illustrates the distribution of stream function $\Psi(r,z)$ for a single vortex ring ($\Gamma=1.0$, $R=0.8$, $Z=0.0$, $n=0.01$ and $R_0=1.0$, $R_C=1.2$, $\Delta_Z=0.3$ and $M=7$). Topographic levels are plotted with discreteness $\Delta\Psi=0.01$. Values of radial velocity induced by the vortex ring are shown in fig.3. Boundary condition for radial velocity component is satisfied.

Thus an evolution of the system of N thin axisymmetrical vortices inside the circular pipe is described by the system of ordinary differential equations of the first order by analogy with the case in unboundary space

$$\frac{dR}{dt}=-\frac{1}{\Gamma_i R_i}\frac{\partial U}{\partial Z_i}, \qquad \frac{dZ_\alpha}{dt}=\frac{\Gamma_\alpha}{4\pi R_\alpha}\left(\ln\frac{8R_\alpha}{a_\alpha}-\frac{1}{4}\right)+\frac{1}{\Gamma_\alpha R_\alpha}\frac{\partial U}{\partial R_\alpha}, \qquad a_\alpha^2 R_\alpha=\mathrm{const}, \qquad (4)$$

where $U=\sum_{\alpha=1}^{N}{\sum_{\gamma=1}^{N}}'\frac{\Gamma_\alpha\Gamma_\gamma}{4\pi}\sqrt{R_\alpha R_\gamma}\left[\left(\frac{2}{k_{\alpha\gamma}}-k_{\alpha\gamma}\right)\mathbf{K}(k_{\alpha\gamma})-\frac{2}{k_{\alpha\gamma}}\mathbf{E}(k_{\alpha\gamma})\right]+\sum_{\alpha=1}^{N}\sum_{m=1}^{M}\frac{\Gamma_\alpha\Gamma_m}{4\pi}\sqrt{R_\alpha R_C}\left[\left(\frac{2}{k_{\alpha m}}-k_{\alpha m}\right)\mathbf{K}(k_{\alpha m})-\frac{2}{k_{\alpha m}}\mathbf{E}(k_{\alpha m})\right]$

with initial conditions: $R_\alpha(0)=R_\alpha^0$, $Z_\alpha(0)=Z_\alpha^0$, $n_\alpha(0)=n_\alpha^0=a_\alpha/R_\alpha$. The prime near summation sing denotes that singular term $\alpha=\beta$ is omitted. The system of equations (4) has two independent invariants

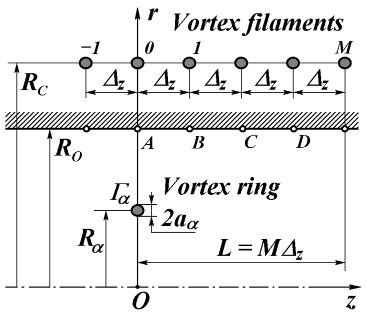

Fig.1. Coaxial vortex rings inside the circular cylinder

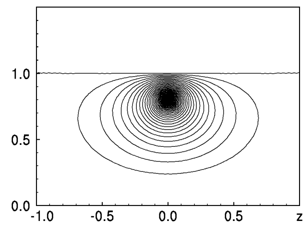

Fig.2. Stream function distribution inside the circular cylinder

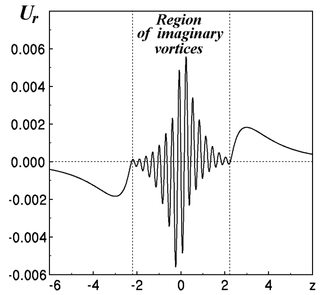

Fig.3. Radial velocity on a surface of the circular cylinder

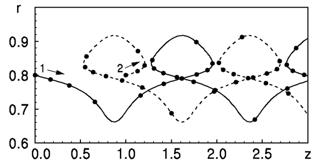

Fig.4. Periodical interaction of two equal vortex rings inside cylinder

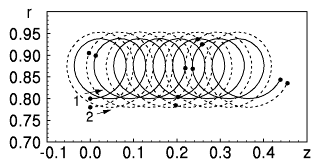

Fig.7 Periodical interaction of two rings with opposite sings of intensity

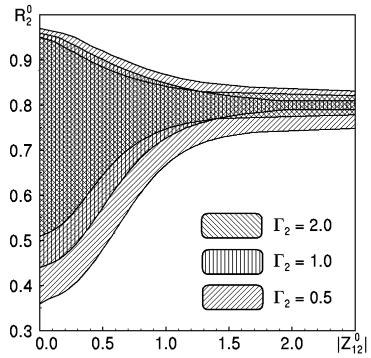

Fig.5. Initial condition of two vortices with the same sing of intensity

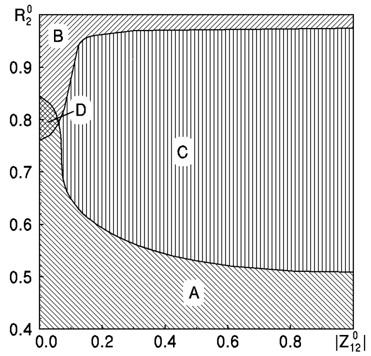

Fig.6. Initial condition of two vortices with the opposite sing of intensity

$$P = \sum_{\alpha=1}^{N} \Gamma_\alpha R_\alpha^2 = \text{const}, \qquad W = \sum_{\alpha=1}^{N} \frac{\Gamma_\alpha^2 R_\alpha}{4\pi} \left(\ln \frac{8R_\alpha}{a_\alpha} - \frac{7}{4} \right) + U = \text{const}, \tag{5}$$

which correspond to the impulse conservation law along axis of symmetry and to the kinetic energy conservation law accordingly.

Researches show that two vortex rings inside cylinder with an ideal incompressible fluid can move stationary. This regime is possible when rings are located in same plane.

The trajectories for periodical interaction of two identical vortex rings ($\Gamma_1 = \Gamma_2 = 1.0$, $R_1^0 = R_2^0 = 0.8$, $Z_1^0 = 0.0$, $Z_2^0 = 0.0$, $n_1^0 = n_2^0 = 0.01$ and $R_0 = 1.0$) are shown in fig.4. Filled circles indicate vortex positions through equal time intervals. When the ring approaches the boundary the influence of boundary increases and vortices even move in opposite direction with respect to self-induced velocity. To identify regions of initial parameters of vortex rings for periodical motion we need to solve a system of nonlinear equations, which contain invariants (5) and condition of equality of self-induced velocity at $|Z_1^0 - Z_2^0| = \infty$ [8]. Fig.5 shows these regions for different values of intensity of the second vortex ring. Analysis shows that two identical vortex rings always participate in a periodical interaction.

To classify types of head-on interaction of two vortex rings we have to solve the same system of equation for $|Z_1^0 - Z_2^0| = 0$ (see details in [7]). Solution of this system equation is shown in fig.6 for $\Gamma_2 / \Gamma_1 = -0.8$. There are four type of interactions: penetration of the smaller ring through a larger ring (A), penetration of the larger ring through a smaller ring (B), head-on collision (C) and periodical interaction (D). An example of periodical interaction ($\Gamma_1 = 1.0$, $\Gamma_2 = -0.8$, $R_1^0 = 0.8$, $R_2^0 = 0.78$, $Z_1^0 = Z_2^0 = 0.0$, $n_1^0 = n_2^0 = 0.01$ and $R_0 = 1.0$) is shown in Figure 7. Rings move at small distances by analogy with periodical interaction of two vortices with opposite signs of their intensities in unboundary space. Analysis of results shows that two identical vertex rings form the closed dynamic system apart from some special cases of interaction.

References

[1] Helmholtz H.: Uber Integrale der hydrodynamischen gleichungen welche den wirbelbewegungen entsprechen. *J. Reine angew. Math.* **B55**:25-55, 1858.

[2] Villat H.: Theorie des tourbillons. Gauthier, Paris 1930.

[3] Prandtl L., Tietjens O. Hydro- and Aeromechanics. Dover, NY 1934.

[4] Meleshko V: Dynamics of vortex structures. Naukova Dumka, Kiev 1993.

[5] Lamb H.: Hydrodynamics. Dover, NY 1945.

[6] Dyson F.W.: The potential of an anchor ring. Pt.II. *Phil. Trans. Roy. Soc. London* **A184**:1041-1106, 1893.

[7] Gourjii A.A.: An interaction of axisymmetrical vortex rings inside endless tube filled by an ideal fluid. *App. Hydromech.* **10** (4):26-42, 2008.

[8] Hicks W.M.: On the mutual threading of vortex rings. *Pro.Roy.Soc.London* **A102**: 111-131, 1922.