APPLICATIONS OF VORTEX DYNAMICS IN MULTIPLY CONNECTED DOMAINS

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Summary The motion of an inviscid, incompressible fluid in a domain consisting of point vortices and three thin plates arranged in a "Kasper wing" formation is considered. Vortex equilibria are sought using a Brownian Rachets scheme and the resulting flow properties of the system for various equilibria are analysed.

PROBLEM SETTING

In many fluid flows involving multiple boundaries the generation of vortices can play an important role in the evolution and characteristics of the flow. For example, the generation of vortices during the flow over an aerofoil can temporarily increase the lift on the aerofoil prior to the vortex being shed. Also, it had been proposed that in some wind turbine configurations the "trapping" of vortices can lead to more efficient energy generation. Thus, in many such scenarios, if the solid boundaries can be arranged in a manner that allows the existence of equilibria involving one or more vortices, the longevity of vortices in such systems could induce beneficial properties on the flow. With this motivation in mind, the motion of an inviscid, incompressible fluid around a theoretical aerofoil design known as the Kasper wing will be considered and stable equilibria involving multiple vortices will be sought using a Brownian Rachets scheme.

The "Kasper Wing", originally proposed by Witroid Kasper in the '70s, is an aerofoil consisting of a main aerofoil and two auxiliary aerofoils designed to increase lift through the trapping vortex structures. Figure 1 shows a schematic of a Kasper wing and the mathematical representation of the aerofoil as a triply connected slit domain.

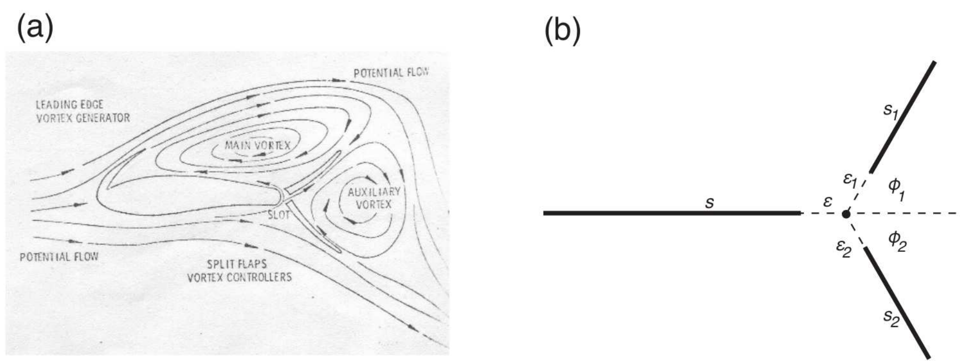

Figure 1. (a) Theoretical design of a Kasper wing. (b) Mathematical representation of the wing as a triply connected domain.

MATHEMATICAL FORMULATION

Functional forms of the equations describing the motion of point vortices in multiply connected circular domains have been derived in Crowdy & Marshall[1]. Let the pre-image domain consisting of the unit disk with two excised circular disks be labeled the $\mathbf{D}_\zeta$ domain and the domain consisting of three slits be the $\mathbf{D}_z$ domain. A schematic of this map is shown in figure 2.

The conformal mapping between these two domains can be written in the form

$$z = g(\zeta) = R\left(\frac{\omega(\zeta,\alpha)\omega(\zeta,\alpha^{-1})}{\omega(\zeta,\beta)\omega(\zeta,\beta^{-1})}\right), \qquad \alpha,\beta,R \in \mathbb{R}, \tag{1}$$

where $\omega(\zeta,\gamma)$ is the Schottky-Klein prime function associated with the circular D_ζ domain (see [1] for more details). Using this conformal mapping and the equations of motion of point vortices in multiply connected circular domains [2], stationary configurations of N vortices of strengths κ_m located at ζ_m $(m = 1,..,N)$ can be obtained through solving

$$\begin{aligned} &\frac{1}{2\pi i}\sum_{k\neq m}^{N}\kappa_k\left(\frac{\omega_\zeta(\zeta_m,\zeta_k)}{\omega(\zeta_m,\zeta_k)} - \frac{\omega_\zeta(\zeta_m,\zeta*_k^{-1})}{\omega(\zeta_m,\zeta*_k^{-1})}\right) - \frac{\kappa_m}{2\pi i}\frac{\omega_\zeta(\zeta_m,\zeta*_m^{-1})}{\omega(\zeta_m,\zeta*_m^{-1})} + i\frac{\kappa_m}{4\pi}\frac{g_{\zeta\zeta}(\zeta_m)}{g_\zeta(\zeta_m)} \\ &+\frac{dW_\infty}{d\zeta}(\zeta_m) + \frac{dW_U}{d\zeta}(\zeta_m) + \frac{dW_\Gamma}{d\zeta}(\zeta_m) = 0, \qquad m = 1,..,N, \end{aligned} \tag{2}$$

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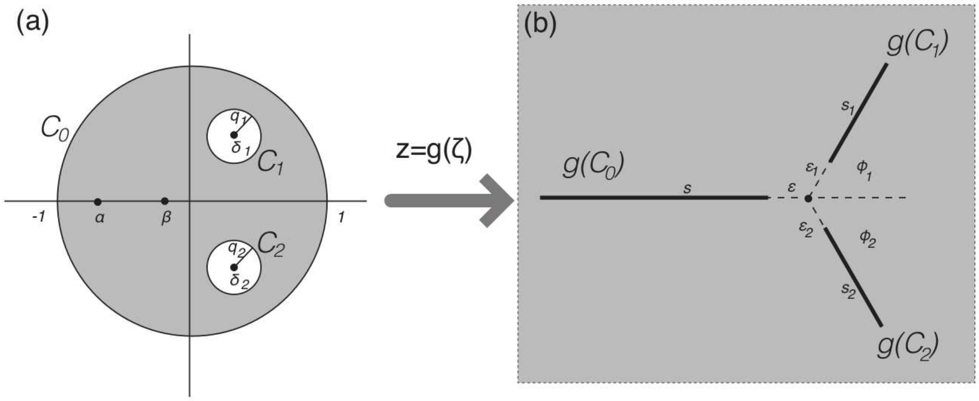

Figure 2. (a) The pre-image $\mathbf{D}_\zeta$ domain (b) The physical $\mathbf{D}_z$ domain. Owing to the Riemann mapping theorem β can be chosen as we wish leaving R, α, q_1 , $q_2 \in \mathbb{R}$ and $\delta_1, \delta_2 \in \mathbb{C}$. In the slit domain there are then also eight real degrees of freedom: $s, \varepsilon, s_1, s_2, \varepsilon_1, \varepsilon_2, \phi_1, \phi_2$ giving rise to a parameter problem that must be solved.

along with the Kutta conditions at the leading and trailing plate edges given by

$$\Im\left(\frac{dW_V}{d\zeta}(\zeta)+\frac{dW_\infty}{d\zeta}(\zeta)+\frac{dW_U}{d\zeta}(\zeta)+\frac{dW_\Gamma}{d\zeta}(\zeta)\right)=0, \qquad \text{at} \quad = l_m, t_m, \qquad m=0,1,2, \tag{3}$$

where l_m and t_m denote the pre-images in the D_ζ domain that are mapped to the leading and trailing edges respectively of the plates in the D_z domain. In the above equations, $W_V(\zeta), W_\infty(\zeta), W_U(\zeta)$ and $W_\Gamma(\zeta)$ denote the complex potentials of the point vortices, the singularities mapped to infinity, the oncoming flow (which is uniform as $|z| \to \infty$) and the round plate circulations respectively. Note that these functions are expressed in terms of Schottky-Klein prime function and parameters of the conformal mapping (1).

METHOD OF SOLUTION

Equations (2) and (3) must be solved simultaneously to determine the stationary point-vortex configurations around the Kasper wing in the presence of the uniform flow. Several approaches are available to solve such systems of equations. Note that (2) and (3) consist of $2N+6$ real equations. First, for a given uniform flow with flux U, the locations of the stationary point vortices give $2N$ real unknowns, the strengths N real unknowns and the round-plate circulations 3 unknowns. Therefore, the number of the equations is equivalent to that of the unknowns only when $N=3$. In this case, the nonlinear $2\times 3+6$ equations could in principle be solved by a Newton method [3]. Another method is a linear algebraic approach called the Brownian ratchet scheme [4]. The basic idea is as follows; (2) and (3) are regarded as a linear equation of the form $Ax=0$, in which A is an $(2N+6)\times(N+4)$ matrix and $x=(U,\kappa_1,\ldots,\kappa_N,\Gamma_0,\Gamma_1,\Gamma_2)^T \in \mathbb{R}^{N+4}$ is a vector to be determined. The matrix A is called the configuration matrix since its components are determined once the locations of the N-point vortices are given. We search for a configuration of the N-point vortices by moving them randomly via Brownian motion until the configuration matrix A has a zero singular value. Then x belongs to its corresponding non-trivial nullspace, giving a one-dimensional solution space. More details are available in [5].

References

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