NONLINEAR MOTION OF CURRENT-VORTEX SHEETS IN MAGNETOHYDRODYNAMIC FLOWS WITH DENSITY STRATIFICATION

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Summary We consider inviscid and incompressible magnetohydrodynamic flows in which current-vortex sheets exist. A current-vortex sheet is defined by a singular surface across which the tangential discontinuities of both the fluid velocity and the magnetic field exist. We derive the equation to describe the temporal evolution of the strength of a current sheet coupled with the evolution equation of the vortex sheet strength, and perform numerical calculations using the vortex blob method. We show that a vortex sheet is deformed by the current that flows on the sheet; especially, motion of the vortex sheet considerably changes due to the polarity (sign) of the current. We present various profiles of the current-vortex sheet which appear in the Rayleigh-Taylor instability (RTI), and show that a current-vortex sheet can stabilize RTI.

A current sheet is defined by a singular surface across which the tangential discontinuity of the magnetic filed exists [2]. Current sheets are important in investigating the large-scale structure of the corona or the solar wind flow in astrophysics and plasma physics [5]. Several studies consider planar and homogeneous current sheets which are nearly equilibrium state [1, 5], in order to be able to treat with analytical methods. When we investigate large deformation of a current sheet such that appears in MHD turbulence, we also need to take into consideration motion of a vortex sheet coexisting with the current sheet. A singular surface across which the tangential discontinuities of both the fluid velocity and the magnetic field exist is defined as a current-vortex sheet [3].

Here, we consider an interface between two fluids with different densities in inviscid and incompressible magnetohydrodynamic flows. The Governing equations in each fluid region i ($i = 1, 2$) are given by

$$\rho_i \left(\frac{\partial \boldsymbol{u}_i}{\partial t} + (\boldsymbol{u}_i \cdot \nabla)\boldsymbol{u}_i + g\hat{\boldsymbol{y}} \right) = -\nabla p_i + \boldsymbol{J}_i \times \boldsymbol{B}_i, \tag{1}$$

$$\frac{\partial \boldsymbol{B}_i}{\partial t} = \nabla \times (\boldsymbol{u}_i \times \boldsymbol{B}_i), \tag{2}$$

$$\nabla \cdot \boldsymbol{u}_i = 0, \tag{3}$$

$$\nabla \cdot \boldsymbol{B}_i = 0, \tag{4}$$

$$\mu_i \boldsymbol{J}_i = \nabla \times \boldsymbol{B}_i, \tag{5}$$

where the constants ρ_i and μ_i are the density and permeability in fluid i ($i = 1, 2$), g is the gravitational acceleration, $\hat{\boldsymbol{y}}$ is the unit vector in the y direction, $\boldsymbol{u}_i$ is the velocity which is related to the velocity potential ϕ_i as $\nabla\phi_i = \boldsymbol{u}_i$, p_i is the pressure, $\boldsymbol{B}_i$ is the magnetic field, $\boldsymbol{J}_i$ is the current density, and we assume that the subscripts 1 and 2 denote the lower and upper fluids of the interface, respectively.

Introducing the magnetic potential χ_i which satisfies $\boldsymbol{B}_i/\sqrt{\mu_i} = \nabla\chi_i$ and imposing the pressure continuous condition $p_1 + B_1^2/(2\mu_1) = p_2 + B_2^2/(2\mu_2)$ at the interface, equation (1) yields the Bernoulli equation in this system:

$$\rho_1 \left[\frac{\partial \phi_1}{\partial t} + \frac{1}{2}(\nabla\phi_1)^2 + gy \right] - \frac{1}{2}(\nabla\chi_1)^2 = \rho_2 \left[\frac{\partial \phi_2}{\partial t} + \frac{1}{2}(\nabla\phi_2)^2 + gy \right] - \frac{1}{2}(\nabla\chi_2)^2, \tag{6}$$

where the spatial derivative is taken at the interface. Defining the circulation Γ and average velocity potential Φ as $\Gamma = \phi_1 - \phi_2$ and $\Phi = (\phi_1 + \phi_2)/2$, we rewite the Bernoulli equation (6) as

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 2A\frac{\mathrm{D}\Phi}{\mathrm{D}t} - A\boldsymbol{q}\cdot\boldsymbol{q} + \frac{A+2\alpha}{4}\boldsymbol{\kappa}\cdot\boldsymbol{\kappa} - \alpha A\boldsymbol{\kappa}\cdot\boldsymbol{q} + 2Agy + \boldsymbol{B}\cdot\boldsymbol{\sigma}, \tag{7}$$

where $\boldsymbol{q} = \nabla\Phi$, $\boldsymbol{\kappa} = \nabla\Gamma$, $\boldsymbol{B} = \nabla(\chi_1 + \chi_2)/2$, $\boldsymbol{\sigma} = \nabla(\chi_1 - \chi_2)$, and $\alpha = \alpha(A)$ ($|\alpha| \le 1$) is a weighting factor such that $\alpha \neq 0$ for $A \neq 0$. Here, we normalized $\boldsymbol{B_i}/\sqrt{\rho_1 + \rho_2} \to \boldsymbol{B_i}$. The derivative $\mathrm{D}/\mathrm{D}t$ is defined as

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \bar{\boldsymbol{u}}\cdot\nabla, \quad \bar{\boldsymbol{u}} = \boldsymbol{q} + \frac{\alpha\boldsymbol{\kappa}}{2},$$

where $\bar{\boldsymbol{u}}$ is the velocity of a fluid particle on the interface. We consider the interface as a vortex sheet and suppose that the interface $\boldsymbol{x} = (x, y)$ is parameterized by a Lagrange parameter e as $x = x(e, t)$ and $y = y(e, t)$. Then the vortex strength $\kappa(e, t)$ is defined as $\kappa = \boldsymbol{\kappa}\cdot\boldsymbol{t} = \nabla\Gamma\cdot\boldsymbol{t} = \partial\Gamma/\partial s = \Gamma_e/s_e$, where the subscript denotes the differentiation with respect to the variable, e here, s is arc length of the sheet, and $\boldsymbol{t}$ is a unit tangent vector of the interface.

Now we define a function χ^- as $\chi^- = \chi_1 - \chi_2$. Then the current density $\boldsymbol{J}$ in a current sheet is given by

$$\boldsymbol{J} = -\boldsymbol{n}\times\nabla\chi^- = (0, 0, J) = \left(0, 0, \frac{\chi_e^-}{s_e}\right) = \frac{\chi_e^-}{s_e}\hat{\boldsymbol{z}} = J\hat{\boldsymbol{z}}, \tag{8}$$

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where $\chi_e^-/s_e = \partial\chi^-/\partial s = \nabla\chi^- \cdot \boldsymbol{t}$. Defining $\boldsymbol{\sigma}$ as $\boldsymbol{\sigma} = \nabla\chi^-$ at the interface, we have

$$\boldsymbol{\sigma} \cdot \boldsymbol{t} = \frac{\partial \chi^-}{\partial s} = \frac{\chi_e^-}{s_e} = J = \sigma. \tag{9}$$

The velocity of the sheet induced by the volticity at the interface is given by the Birkhoff-Rott equation [4]

$$\frac{\partial z^*(e,t)}{\partial t} = \frac{1}{2\pi i}\mathrm{PV}\int \frac{\kappa(e',t)s_e(e')de'}{z(e,t)-z(e',t)} + \frac{\alpha\kappa(e,t)z_e^*(e,t)}{2s_e(e)}, \tag{10}$$

where PV denotes Cauchy's principal value integral, $z(e,t) = x(e,t) + iy(e,t)$ and z^* is the complex conjugate of z. This equation corresponds to the Biot-Savrt law in 2D. Separating the real and the imaginary parts in equation (10), we obtain

$$\frac{\partial x}{\partial t} = U + \frac{\alpha x_e}{2s_e}\kappa, \quad \frac{\partial y}{\partial t} = V + \frac{\alpha y_e}{2s_e}\kappa, \tag{11}$$

in which

$$\begin{aligned} U(e,t) &= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sinh\left(y(e,t)-y(e',t)\right)\kappa(e',t)s_e(e')de'}{\cosh\left(y(e,t)-y(e',t)\right) - \cos\left(x(e,t)-x(e',t)\right) + \delta^2}, \\ V(e,t) &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sin\left(x(e,t)-x(e',t)\right)\kappa(e',t)s_e(e')de'}{\cosh\left(y(e,t)-y(e',t)\right) - \cos\left(x(e,t)-x(e',t)\right) + \delta^2}, \end{aligned} \tag{12}$$

where we assumed the periodicity of the vortex sheet. The parameter δ was introduced by Krasny in order to regularize the Cauchy integral. This approach is called the vortex blob method [4].
Differentiating equation (7) with respect to e, we obtain the following Fredholm integral equation of the second kind:

$$\begin{aligned} \frac{\mathrm{D}\kappa}{\mathrm{D}t} &= \frac{2A}{s_e}\left(x_e\frac{\partial U}{\partial t} + y_e\frac{\partial V}{\partial t}\right) - \frac{(1+\alpha A)\kappa}{s_e^2}(x_eU_e + y_eV_e) \\ &+ \frac{A+\alpha}{4s_e}(\kappa^2)_e + 2Ag\frac{y_e}{s_e} + \left[\frac{\sigma(x_eB_x + y_eB_y)}{s_e}\right]_e, \end{aligned} \tag{13}$$

where we set $\boldsymbol{B} = (B_x, B_y)$. The magnetic induction equation (2) on the interface is given as

$$\begin{aligned} \frac{\mathrm{D}\sigma}{\mathrm{D}t} &= -\frac{\sigma}{s_e}\left(\frac{U_ex_e + V_ey_e}{s_e} + \frac{\alpha}{2}\kappa_e\right) + \left[\frac{\alpha}{2}\sigma_e - \left(\frac{B_xx_e + B_yy_e}{s_e}\right)_e\right]\kappa \\ &+ \frac{B_xx_e + B_yy_e}{s_e}\kappa_e + \sigma\left(\frac{U_ex_e + V_ey_e}{s_e}\right)_e. \end{aligned} \tag{14}$$

The average magnetic field on the sheet $\boldsymbol{B} = (B_x, B_y)$ is given by the Biot-Savart law in the similar manner to equation (12):

$$\begin{aligned} B_x(e,t) &= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sinh\left(y(e,t)-y(e',t)\right)\sigma(e',t)s_e(e')de'}{\cosh\left(y(e,t)-y(e',t)\right) - \cos\left(x(e,t)-x(e',t)\right) + \delta^2}, \\ B_y(e,t) &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sin\left(x(e,t)-x(e',t)\right)\sigma(e',t)s_e(e')de'}{\cosh\left(y(e,t)-y(e',t)\right) - \cos\left(x(e,t)-x(e',t)\right) + \delta^2}. \end{aligned} \tag{15}$$

Solving equations (11), (13), and (14) simultaneously, we can determine motion of a current-vortex sheet. From the linear stability analysis, we can show that RTI can be stabilized due to the existence of the magnetic field. We will also present the results of numerical calculations of the governing equations.

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