EFFECTS OF AXIAL FLOW ON THE CURVATURE INSTABILITY OF A HELICAL VORTEX TUBE: MODAL ANALYSIS

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Summary The short-wave instability of a helical vortex tube with axial flow is studied by modal stability analysis. The exponential growth rate is evaluated by perturbation expansion, where the expansion parameter is the ratio of core to curvature radius. At the first order the instability is a superposition of the curvature and precession instabilities. The combined effects of torsion and axial flow appear at the second order. The results converge to those obtained by local stability analysis in the short-wave limit.

INTRODUCTION

Helical vortices are formed around rotating blades which often appear in engineering applications including helicopter rotors, wind rotors and turbine impellers. Since the flow field and the performance of the wings or the devices are affected by the helical vortices, it is important to understand their dynamical properties among which one of the most fundamental is stability.

There are three types of instability of helical vortices: the long-wave instability, the short-wave instability and the mutual inductance mode[1]. We focus on the short-wave instability for which the wavenumbers of unstable modes are comparable to the core size of the vortices since it is less understood than the other two types. We have studied the stability of a helical vortex tube without axial flow which is a model of the helical vortices[2]. By local stability analysis the helical vortex tube was shown to be subject to curvature instability which had been only known for vortex rings[3, 4]. A helical vortex is featured by the presence of torsion in its configuration, and its motion comprises translation and rotation, without change of form, about an axis. The effects of torsion and rotation on the curvature instability were also investigated. Recently we have extended our previous analysis to a helical vortex tube with axial flow[5]. The effects of the axial flow are explored in some detail. However, the local stability analysis is limited to the case of large wavenumber.

In this paper we investigate the effects of the axial flow on the curvature instability of a helical vortex tube by modal stability analysis. Our aim is to obtain the complete picture of the curvature instability of the helical vortex tube by analyzing the case of wavelength comparable to or larger than the core size.

BASE FLOW OF HELICAL VORTEX TUBE WITH AXIAL FLOW

We consider a helical vortex tube with axial flow whose centerline is a helix of constant curvature and torsion, coiling on the surface of a cylinder, as shown in Fig. 1. We assume the flow be incompressible and inviscid. The vortex tube, as a whole, translates along and rotates with angular velocity about the axis of the cylinder. The core radius of the tube is assumed to be small but finite. The base flow of the vortex tube with axial flow is obtained by perturbation expansion; the expansion parameter ϵ is defined as the ratio of core to curvature radius of the helical tube[5]. The leading-order flow field is set to rigid body rotation with uniform axial flow. Then the flow field turns out to be

$$\boldsymbol{U} = \begin{pmatrix} 0 \\ r \\ \mu \end{pmatrix} + \epsilon \begin{pmatrix} \frac{5}{8}\left(1-r^2\right)\sin\varphi \\ \left(\frac{5}{8}-\frac{7}{8}r^2\right)\cos\varphi \\ \mu r\cos\varphi \end{pmatrix} + \epsilon^2 \begin{pmatrix} \left(\frac{1}{8}r-\frac{1}{8}r^3\right)\sin 2\varphi + \alpha\mu\left(\frac{7}{8}-\frac{7}{8}r^2\right)\sin\varphi \\ \left(\frac{1}{8}r-\frac{1}{16}r^3\right)\cos 2\varphi + \alpha\mu\left(\frac{7}{8}-\frac{13}{8}r^2\right)\cos\varphi \\ \left[\left(\frac{5}{8}\alpha+2\gamma\right)r-\frac{3}{8}\alpha r^3\right]\cos\varphi+\frac{\mu}{2}r^2\cos 2\varphi \end{pmatrix} + O(\epsilon^3), \tag{1}$$

in the orthogonal coordinate system (r, θ, s) along the helical tube. Here μ is the uniform axial flow velocity at the leading order, α is the torsion, γ is the rate of rotation and $\varphi = \theta - \epsilon\alpha s$.

MODAL STABILITY ANALYSIS

The stability of the helical vortex tube is studied by modal analysis. The linearized Euler equations in a rotating frame

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{u}' + \boldsymbol{u}'\cdot\nabla\boldsymbol{U} + 2\boldsymbol{\Omega}\times\boldsymbol{u}' = -\nabla p', \quad \nabla\cdot\boldsymbol{u}' = 0, \tag{2}$$

are expressed in the present coordinate system by perturbation expansion. Since the base flow field above induces parametric resonance at $O(\epsilon)$ by terms proportional to $e^{\pm i\varphi}$, the disturbance is set to

$$\boldsymbol{u'}_0 = A\exp\left[\mathrm{i}\left(-\omega t + m\varphi + ks\right)\right]\boldsymbol{u'}_{0,m} + B\exp\left[\mathrm{i}\left(-\omega t + (m+1)\varphi + ks\right)\right]\boldsymbol{u'}_{0,m+1}, \tag{3}$$

at the leading order. The leading-order equations can be expressed as

$$L_0(\omega, k-\epsilon m\alpha, m)\boldsymbol{u'}_{0,m} = 0, \quad L_0(\omega, k-\epsilon(m+1)\alpha, m+1)\boldsymbol{u'}_{0,m+1} = 0, \tag{4}$$

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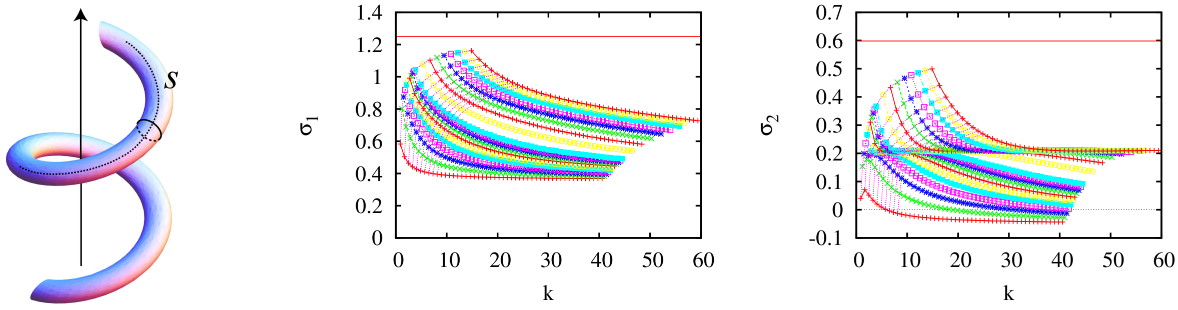

Figure 1. (Left) Helical vortex tube. (Middle) First-order growth rate. (Right) Second-order growth rate. $\alpha = \mu = 1, \gamma = 0$.

where L_0 is a linear operator. In order to obtain non-trivial solutions the operator L_0 should be singular; this condition leads to dispersion relations according to which ω is real so that the base flow is neutrally stable at the leading order. At $O(\epsilon)$ the terms due to resonance appear in the equations

$$L_0(\omega, k - \epsilon m\alpha, m)\boldsymbol{u'}_{1,m} = BN_1^{-1}\boldsymbol{u'}_{0,m+1} - \frac{\mathrm{d}A}{\mathrm{d}t_1} I_v \boldsymbol{u'}_{0,m}, \tag{5}$$

$$L_0(\omega, k - \epsilon(m+1)\alpha, m+1)\boldsymbol{u'}_{1,m+1} = AN_1^{+1}\boldsymbol{u'}_{0,m} - \frac{\mathrm{d}B}{\mathrm{d}t_1} I_v \boldsymbol{u'}_{0,m+1}, \tag{6}$$

where $N_1^{\pm}$ denotes the operator due to $O(\epsilon)$ base flow field and the amplitudes A and B should vary slowly with time scale $t_1 = \epsilon t$ to satisfy compatibility condition for the singular operators. Similar equations are obtained at $O(\epsilon^2)$. Taking the inner product defined by $< \boldsymbol{u}_a | \boldsymbol{u}_b >= \int_0^1 (\overline{u}_a u_b + \overline{v}_a v_b + \overline{w}_a w_b + \overline{p}_a p_b) r \; dr$ we obtain the following amplitude equations

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \mathrm{i}(\epsilon a_1 + \epsilon^2 a_2)B + \mathrm{i}\epsilon^2 c_2 A, \quad \frac{\mathrm{d}B}{\mathrm{d}t} = \mathrm{i}(\epsilon b_1 + \epsilon^2 b_2)A + \mathrm{i}\epsilon^2 d_2 B, \tag{7}$$

where the coefficients are expressed as $\mathrm{i}a_1 = \langle \boldsymbol{u'}_{0,m} | N_1^{-1} \boldsymbol{u'}_{0,m+1} \rangle / \langle \boldsymbol{u'}_{0,m} | I_v \boldsymbol{u'}_{0,m} \rangle$ for example. Finally the growth rate is

$$\sigma = \epsilon \left(-a_1 b_1\right)^{1/2} \left[1 + \frac{\epsilon}{2}\left(\frac{a_2}{a_1} + \frac{b_2}{b_1}\right)\right] + O(\epsilon^3). \tag{8}$$

The growth rate is evaluated numerically for principal modes of $0 \le m \le 50$ (Fig. 1). Also shown in the figure are the values obtained by local stability analysis for the radius of the fluid particle motion $r_0 = 1$ and the angle of wavevector $\phi = \pi/2$[5]

$$\sigma = \epsilon \left(\frac{165}{256} + \frac{5\sqrt{15}\mu}{32}\right) + \epsilon^2 \left(\frac{5\sqrt{15}\gamma}{32} - \frac{\sqrt{15}\alpha\mu^2}{48} - \frac{3\sqrt{15}\alpha}{128} + \frac{197}{256}\alpha\mu\right). \tag{9}$$

Both the first and second order terms converge to (9) in the limit $m \sim k \to \infty$. The first-order growth rate is the sum of terms due to curvature and axial flow. At $O(\epsilon)$ the base flow is essentially that of a vortex ring as the effects of torsion appear at $O(\epsilon^2)$. Since the uniform axial flow of a vortex ring is identical with rotation about the axis of symmetry, it induces the Coriolis or precession instability. The second-order growth rate is positive, or in other words, the total growth rate increases for $\alpha\mu > 0$ except for large k and small m.

CONCLUSIONS

The instability of a helical vortex tube with axial flow is studied by modal stability analysis. At $O(\epsilon)$ the instability is a superposition of the curvature and the precession instabilities. At $O(\epsilon^2)$ combined effects of torsion and axial flow appear. In the present analysis the boundary of the core is fixed and the slip boundary conditions are imposed. More plausible case of free boundary will be also considered.

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