STATIONARY VORTEX STATES IN A TWO-LAYER ROTATING FLUID

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Summary New stationary solutions of the system of three and four discrete vortices in a two-layer rotating fluid are proposed. A qualitative analysis of the relative motion of vortices is given, which allows a study of the stability of stationary states.

MODEL

The equations of motion of discrete vortices, in a two-layer rotating fluid were first derived by Gryanik [3]. The upper and lower fluid layers have non-disturbed thicknesses h_1 and h_2 and constant densities ρ_1 and ρ_2.These equations were then used in numerous investigations, particularly in [6]–[9].In a particular case, when the number of vortices is equal to three, phase portraits of a corresponding dynamical system (and, therefore, all relative motions of vortices) can be studied exhaustively by applying the so called trilinear coordinates[1]. This method can be generalized to the case of four vortices under the condition that the total intensity and impulses are zero [2]. Let us note that this analysis doesn't depend on the type of Hamiltonian system and so, can be extended to a model of two-layer ocean [7].
Let variables A_1 and A_2 be the number of vortices in the upper and bottom layers, correspondingly. We will examine the cases $A_1 = 1$, $A_2 = 2$ and $A_1 = A_2 = 2$.Let us limit ourselves to the cases of zero total intensity, i.e. in the first case $\kappa_1^1 = -2\kappa$, $\kappa_2^1 = \kappa_2^2 = \kappa$; in the second case $\kappa_1^1 = -\kappa_1^2 = \kappa$, $\kappa_2^1 = -\kappa_2^2 = -\kappa$ (then, applying the terminology of [4], we have two *hetons* – cold and hot ones) or $\kappa_1^1 = \kappa_1^2 = -\kappa$, $\kappa_2^1 = \kappa_2^2 = \kappa$ (two hot *hetons*).

RESULTS

Three corresponding variants of initial collinear states of the three point-vortex system are shown in figure 1; figure 2 gives phase portraits in trilinear coordinates for these three cases. Let us note that the vortex configurations shown in figure1 do not cover all possible initial states of the system, as there are variants when there is no collinear position (for example, motions of the type $\{4\}$ in figure 2b and of type $\{1\}$ in figure 1c). However, we are interested here only in stationary solutions, which correspond to elliptical special points in the areas $\{2\}$ in figure 2a and in the area $\{3\}$ in figure 2b, and also on the dashed isoline in Figure 2c and to hyperbolic points in all three panels.

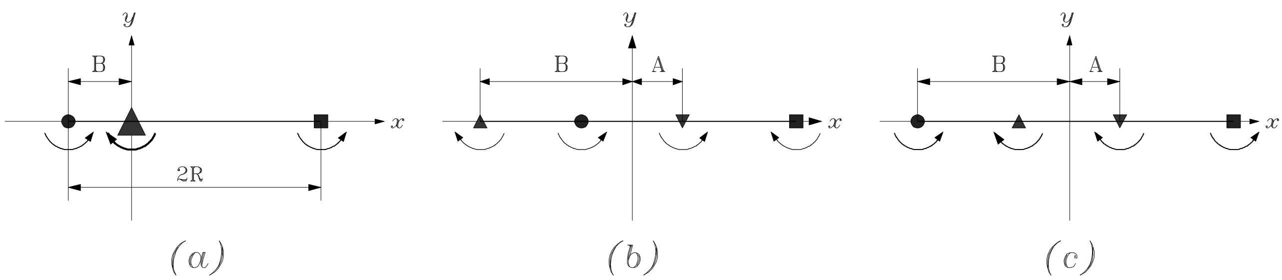

Figure 1. General layout of the initial collinear position of discrete vortices for cases (a) $A_1 = 1$, $A_2 = 2$, $\kappa_1^1 = -2\kappa$, $\kappa_2^1 = \kappa_2^2 = \kappa$; (b) $A_1 = A_2 = 2$, $\kappa_1^1 = -\kappa_1^2 = \kappa$, $\kappa_2^1 = -\kappa_2^2 = -\kappa$; (c) $A_1 = A_2 = 2$, $\kappa_1^1 = \kappa_1^2 = -\kappa$, $\kappa_2^1 = \kappa_2^2 = \kappa$. Triangle markers correspond to the vortices of the upper layer, circles and squares to the vortices of the bottom layer. The size of the markers is proportional to the modulus of intensity. Arc arrows show the direction of rotation ($\kappa > 0$).

Solutions representqtive of the non-symmetrical configuration of three vortices with an anticyclonic upper layer vortex inside and two cyclonic vortices at the periphery, in solid body rotation with constant positive angular velocity, correspond to the elliptical special points in the areas $\{2\}$ in figure 2a. We named such a vortex construction a *triton* ([8]–[9]).
Collinear four-vortex constructions, belonging in turn to the lower and upper layers, and rotating with constant angular velocity with respect to the geometrical center [6] correspond to the elliptical special point in the areas $\{3\}$ in figure 2b. This is a particular case of so called *heton quartets* [5]. As figure 2 shows, both these solutions, represented by elliptical points in the phase plane, are stable.
Unstable stationary states are: (a) under the configuration of an isosceles triangle, moving forward parallel to the base line, with an anticyclonic upper-layer vortex in the vertex of a triangle and two bottom-layer vortices at its base angles (hyperbolic special point in figure 2a); (b) uniformly rotating construction of four vortices shaped as a parallelogram and diamond (the same in figures 2b and 2c, correspondingly). Because of their instability, these solutions cannot be achieved numerically.

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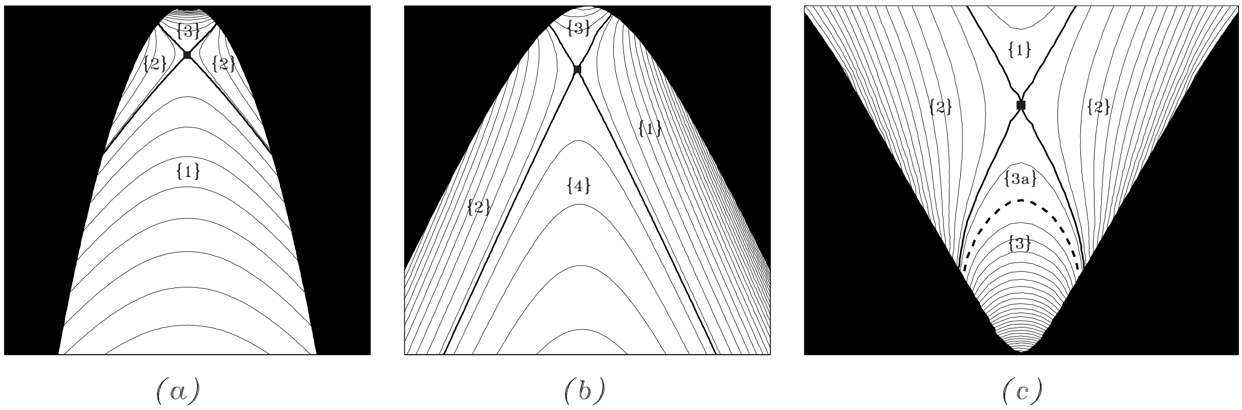

Figure 2. Phase portraits in trilinear coordinates for the system of point vortices are given in diagram form in panels (a), (b) and (c) of the figure 1, correspondingly. Solid lines mark separatrices, dividing areas with different motion types $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$. Square markers designate everwhere hyperbolic points of separatrix crossing.

A family of vortex structures, shown in figure 3b, represents non-trivial stationary states: two vortices of one of the layers rotating along the common quasi-elliptical trajectory, and two vortices of the other layer perform nutational rotation along individual drop-shaped peripheral trajectories. Rotation frequencies of the different-layer vortices make a ratio of 1:2. This construction keeps its shape in the fixed coordinate system. If the parameters A and B deviate in either direction, invariability of the construction remains only in the rotating coordinate system – clockwise (type $\{3\}$) or counterclockwise (type $\{3a\}$). Another possible scenario is a transition into the area $\{2\}$ where the trajectiries scatter (two hetons diverge).

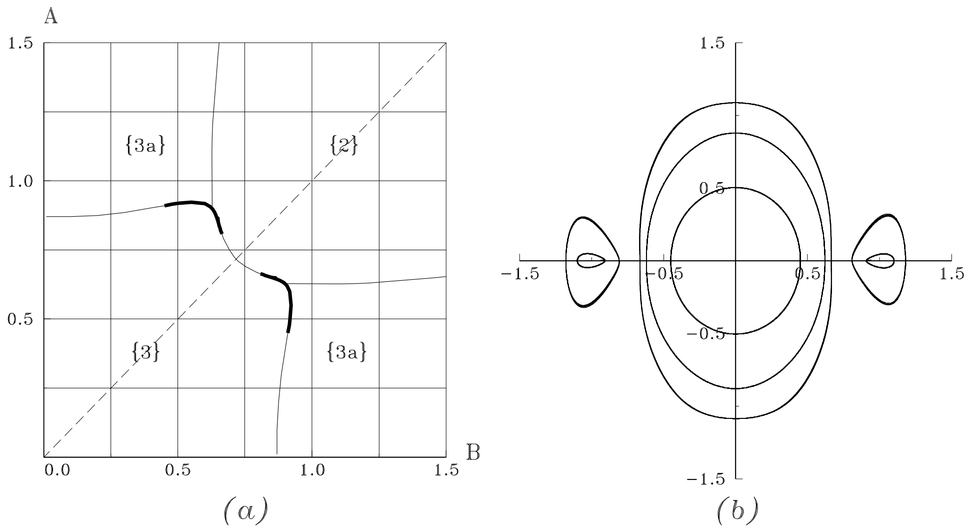

Figure 3. (a) State diagram of two hot-heton system, initially forming a collinear configuration (figure 1c) in the plane of parameters (B, A). (b) A set of quasistationary trajectories at $(B, A) = (0.6640, 0.8075)$, $(B, A) = (0.6200, 0.9069)$ and $(B, A) = (0.4500, 0.9104)$. These values of the parameters A and B belong to the solid-line areas of separating lines of the panel (a), which are the analogs of the dashed line in figure 2c.

Let us note that study of stationary states has fundamental importance in the vortex dynamics.

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