TRACKING OF SPATIALLY LOCALIZED SOLUTIONS BY A FILTERING METHOD: APPLICATION TO THE SWIFT-HOHENBERG EQUATION

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Summary Spatially localized solutions have played key roles in understanding turbulence in pipe and other flows. So far, however, there seems to be no general technique to find out such localized solutions. We have introduced a filtering method successfully. Here, we apply this method to the Swift-Hohenberg equation having homoclinic snaking that is found even in Couette flow and show that a closed branch of localized solutions with internal structure is obtained from a solution on the snaking curve.

INTRODUCTION

The spatially localized exact solutions such as travelling-waves to the Navier-Stokes equation have played key roles in understanding turbulence in pipe, wall-bounded shear flow and other flows. Each of these exact solutions has discovered by heuristic methods. In this sense, their applications are limited to specific cases.
To find out the spatially localized exact solutions in various cases, we have introduced a filtering method [1] by which we have succeeded to obtain localized solutions to one-dimensional Swift-Hohenberg equation(SHE) and channel flows. In this study, we apply the filtering method to one-dimensional SHE and show that the method works well in finding out nontrivial solutions.

FILTERING METHOD

We add the filtering term $-A_f H(x)u$ to the r.h.s. of SHE as follows:

$$\begin{aligned} u_t &= -\frac{\delta F}{\delta u} - A_f H(x)u, \\ F[u] &= \int_0^L dx \left[-\frac{1}{2} r u^2 + \frac{1}{2}\left(\left(\frac{\partial^2}{\partial x^2} + q_c^2 \right) u \right)^2 - \frac{b_3}{4}u^4 + \frac{b_5}{6}u^6 \right], \end{aligned} \tag{1}$$

where $u = u(x,t), x \in [0, L]$, and the fixed boundary condition $u(0) = u(L) = 0$ is used. The filter function is defined by the error function: $H(x; \alpha, \beta) = 1 - [\mathrm{erf}((x - (L/2 - \alpha))/2\beta) + \mathrm{erf}((x - (L/2 + \alpha))/2\beta)]/2$ where α and β are the parameters determining the width of the filter. The Lyapunov function $F[u]$ is called the energy hereafter for simplicity.
To obtain a localized steady solution, we integrate Eq.(1) in time with a positive filter amplitude $A_f > 0$. Starting with this solution, we track the solution by the Newton-Krylov algorithm with changing A_f. When A_f gets to 0, a solution of SHE is obtained. Note that A_f can be negative in the tracking as shown in the followings. Owing to the Krylov subspace method, we can reduce the computational cost drastically.

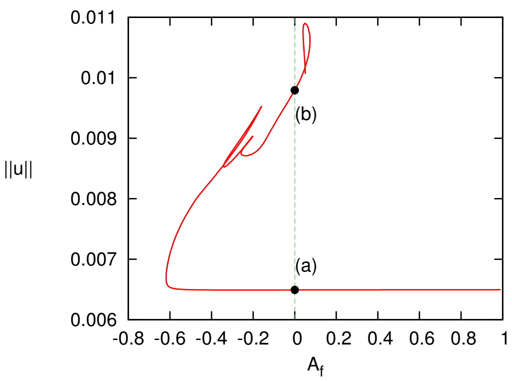

Figure 1. The L^2-norm $||u||$ of spatially localized steady solutions of Eq.(1) is plotted against the filter amplitude A_f. The points labeled by (a) and (b) at which the branch across the $A_f = 0$ represent solutions of original SHE.

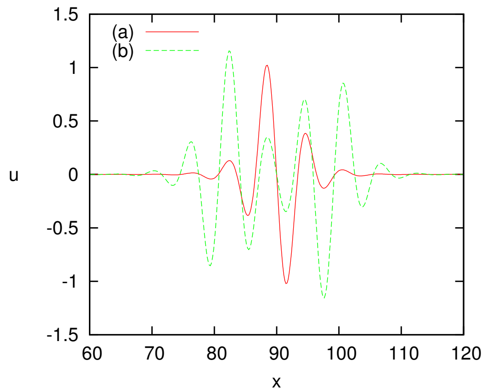

Figure 2. The profiles of the two solutions of original SHE: red line,the solution (a); green dashed line, (b) shown in Fig.1.

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RESULTS

In Fig.1, we show a branch of localized solutions in the 2D parameter space spanned by A_f and L_2-norm, $||u|| = (\int_0^L dx[u(x)^2])^{\frac{1}{2}}$. In this case, we obtain two exact solutions (a) and (b) to the original SHE. The profiles of the solutions (a) and (b) are plotted in Fig.2. The solution (b) is also a homoclinic solution connecting the homogeneous solution $u(x) = 0$. Note that it is not a simple heteroclinic connection between the homogeneous and periodic solutions but has internal structure.

Next, to make clear the meaning of these solutions in the original SHE, we track these solutions, in turn, for the parameter r in the same way done for A_f. The continuations indicate that each of these solutions belongs to the different solution branch as shown in Fig.3. The solution (a) is on so-called the snaking branch (Fig.3,A) in which the number of peak and valley of the solution increases by one in every snaking [2].

On the other hand, the branch containing the solution (b) is a closed and isolated one and also seems to be reported first. Thus the detection of this branch might be a difficult task in the use of the native parameters.

Figure 4 shows the comparison of the energy (Lyapunov function) F of the solutions on the two branches having the same value of the L^2-norm. The solutions on the branch B have lager energy than those on the branch A. Since the energy F decreases monotonically, the solutions on the branch B are thought to be in one of excited states so that they have more complicated internal structures than those on the snaking branch.

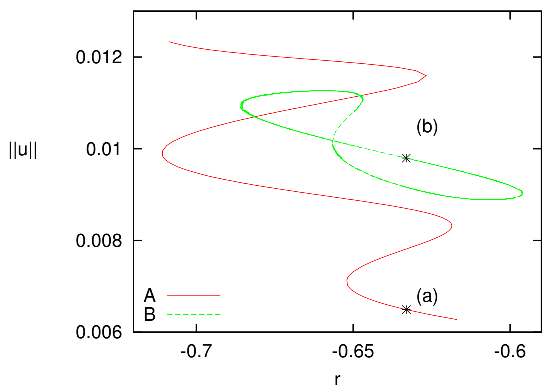

Figure 3. The norms $||u||$ of the two branches of original SHE ($A_f = 0$) are plotted against the parameter r. The two points labeled by (a) and (b) represent the same solutions in Fig.1. Branch A (red line) denotes the continued solutions from (a), and Branch B (green dashed line) from (b).

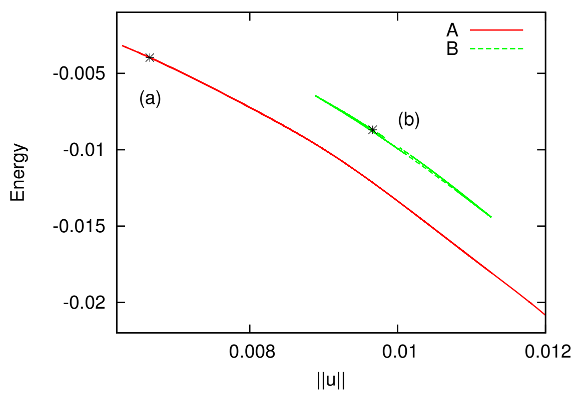

Figure 4. Comparison of the Energy (Lyapunov function) F of the solutions with the same norm $||u||$ on the branches A and B: red line, branch A; green dashed line, branch B . The solutions (a) and (b) in Fig.3 are also plotted.

CONCLUDING REMARKS

We have obtained a new closed and isolated branch that is connected to the snaking branch in the extended solution space. The solutions on this branch have energy larger than those on the snaking branch. This means that they have internal structures.

Our results suggest that connections between spatially localized solutions in extended parameter space could exist even for equations with localized solutions similar to those of SHE. For example, in plane Couette flow snaking structure, homoclinic snaking structure, was reported [3]. Moreover, we expect that our filtering method could be applicable to more general cases such as "large scale motion" observed in wall-bounded flows at high Reynolds number.

References

[1] still under writing.
[2] J. Burke and E. Knobloch, "Homoclinic snaking: Structure and stability" *Chaos* **17** 037102(2007)
[3] T. M. Schneider, J.F. Gibson, and J.Burke, "Snakes and Ladders: Localized Solutions of Plane Couette Flow" *Phys. Rev. Lett.* **104** 104501(2010)