A MULTI-LAYER MODEL FOR TURBULENT KINETIC ENERGY IN PIPE FLOWS

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Summary A multi-layer model of a kinetic energy length is developed by employing recent results of the authors [1]. The theory predicts the complete, mean streamwise turbulent kinetic-energy profile (MKP), in good agreement with empirical data for a wide range of the Reynolds number (*Re*). In particular, a critical Re_τ around 5000 is predicted, beyond which a scaling anomaly appears and MKP develops a second peak whose location varies as $\mathrm{Re}_\tau^{1/2}/\kappa$ (κ is the Karman constant) and kinetic energy peak value as $3.5\,\mathrm{Re}_\tau^{0.06}$.

INTRODUCTION

Development of an analytical theory for the prediction of turbulent mean velocity profile (MVP) and MKP for the entire pipe flow has persisted as an elusive goal for over a century. Recently, notable progress has been made in the former, but far less in the latter. An interesting observation in MKP is the second kinetic energy peak located in the log layer at high *Re* [2], which signal a new outer structure, but not revealed by the existing models [3]. To describe such an outer peak, a centerline velocity scaling is proposed [4]; however, a complete solution for MKP is still missing.

MULTI-LAYER MODEL FOR LENGTH ORDER FUNCTIONS

We extend a newly developed statistical mean-field theory based on the symmetries of "*order parameter*", which are defined, for wall-bounded turbulent flows, as a set of characteristic lengths. The mixing length function is $\ell_m^+ = \sqrt{-\langle uv\rangle^+}/S^+$, where the mean shear $S^+ = dU^+/dy^+$, Reynolds stress $W^+ = -\langle uv\rangle^+$, and + denotes wall units normalization; ℓ_m^+ displays a set of distinct dilation invariance in each of the commonly known layers such as viscous sublayer, buffer layer, bulk zone and core zone, and is derived as

$$\ell_m^+ = 0.967\left(\frac{y^+}{9.8}\right)^{3/2}\left(1+\left(\frac{y^+}{9.8}\right)^4\right)^{1/8}\left(1+\left(\frac{y^+}{44.7}\right)^4\right)^{-1/4}\frac{1-r^5}{5(1-r)Z_c}\left(1+\left(\frac{0.67}{r}\right)^2\right)^{1/4}, \qquad (1)$$

where variables, y^+ and r, denote the distance to the wall and to the center, respectively, with $Z_c = (1 + 0.67^2)^{1/4} \approx 1.097$. This four-layer model was shown to give an accurate description for MVP over a wide range of *Re* (for $\mathrm{Re}_\tau > 5000$) in Princeton pipe. Here, we introduce an 'energy length' ℓ_u^+, relating the streamwise kinetic energy to shear, i.e. $\ell_u^+ = \sqrt{\langle uu\rangle^+}/S^+$. In addition to all ℓ_m *layers*, ℓ_u^+ here discovers and quantifies a new layer adjacent to the buffer layer, which is indeed the so called meso-layer [5]. The multi-layer formula for ℓ_u^+ is

$$\ell_u^+ = \rho_u y^+\left(1+\left(\frac{y^+}{y^+_{sub}}\right)^3\right)^{\gamma_1/3}\left(1+\left(\frac{y^+}{y^+_{buf}}\right)^4\right)^{\gamma_2/4}\left(1+\left(\frac{y^+}{y^+_{meso}}\right)^2\right)^{\gamma_3/2}\frac{1-r^5}{5(1-r)\tilde{Z}_c}\left(1+\left(\frac{\tilde{r}_c}{r}\right)^2\right)^{1/2} \qquad (2)$$

which leads to an analytic solution for MKP by substitution of S^+ in terms of the mixing length:

$$\langle uu\rangle^+ = \left(\ell_u^+\left(-1+\sqrt{4r\ell_m^{+2}+1}\right)/\left(2\ell_m^{+2}\right)\right)^2. \qquad (3)$$

Here, we introduce a concept of scaling anomaly: the case of $\Delta\gamma = \gamma_1+\gamma_2+\gamma_3 = 0$ corresponds to normal scaling, in which the inner and outer solutions for ℓ_u^+ are both linear in y; while a scaling anomaly appears if $\Delta\gamma \neq 0$. In the later case, a residue scaling developed from inner motion requires further treatment. In the following, we focus primarily on the determination of scaling and thickness parameters, and also give physical interpretation of the two cases.

NORMAL SCALING AT MODERATE *Re* – APPLICATION TO DNS DATA

We adopt the parameter values previously determined [1]: $y^+_{sub} \approx 9.8$, $y^+_{buf} \approx 40.5$, $r_c \approx 0.27$ while $\rho_u \approx 0.42$, $\gamma_1 \approx 2/3$, $\gamma_2 \approx -13/15$, $\gamma_3 \approx 1/5$, $y^+_{meso} \approx 150$ and $\tilde{r}_c \approx 0.23$ are systematically measured from DNS data. The predicted MKP agrees very well with DNS data (see Fig.1 left).

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ANOMALY SCALING AT HIGH *Re* – APPLICATION TO EXPERIMENTAL DATA

As the second peak (at y_p^+) is located in the overlap region, we can derive $y_p^+ \approx y_{meso}^+\sqrt{\gamma_3/\Delta\gamma - 1}$ (for $\Delta\gamma \neq 0$). Now, we make three assumptions which lead to accurate description of the data. The first is $y_p^+ = y_{meso}^+$, meaning that the two geometrical quantities must equal since they describe the only characteristic thickness in this region, which leads to $\gamma_3 = 2\Delta\gamma$. The second is a finite $\langle uu \rangle^+$ at the centerline for large Re, which predicts $y_{meso}^+ \sim \mathrm{Re}_\tau^{\Delta\gamma/\gamma_3} \sim \mathrm{Re}_\tau^{1/2}$. The third is $\ell_m^+(y_p^+) = \sqrt{\mathrm{Re}_\tau}$, motivated from the following estimate. Since $W^+ \approx 1 - y^+/\mathrm{Re}_\tau - 1/(\kappa y^+)$ in the overlap region, we obtain a rigorous estimate of the peak location of W$^+$: $y_w^+ = \sqrt{\mathrm{Re}_\tau/\kappa}$ and $\ell_m(y_w^+) = \sqrt{\kappa \mathrm{Re}_\tau}$. Note that both y_p^+ and y_w^+ scale with $\sqrt{\mathrm{Re}_\tau}$; with the third assumption, we derive the outer peak location as:

$$y_p^+ = y_{meso}^+ = \sqrt{\mathrm{Re}_\tau}/\kappa. \qquad (4)$$

Furthermore, using $\ell_u/(1-r^m) \sim y^{\Delta\gamma}$ for $y^+ \gg y_{meso}^+$, we obtain $\Delta\gamma \approx -0.06$ from data. Thus, $\gamma_3 \approx -0.12$, $\gamma_2 = -(\gamma_1 + \gamma_3/2) \approx -0.607$. Finally, the core layer thickness $\tilde{r}_c \approx 0.37$ is determined from the measured centerline kinetic energy. Using the above values, we have a good description for MKP, and a prediction of the second kinetic energy peak :

$$\langle uu \rangle_p^+ \approx 3.5\,\mathrm{Re}_\tau^{0.06}. \qquad (5)$$

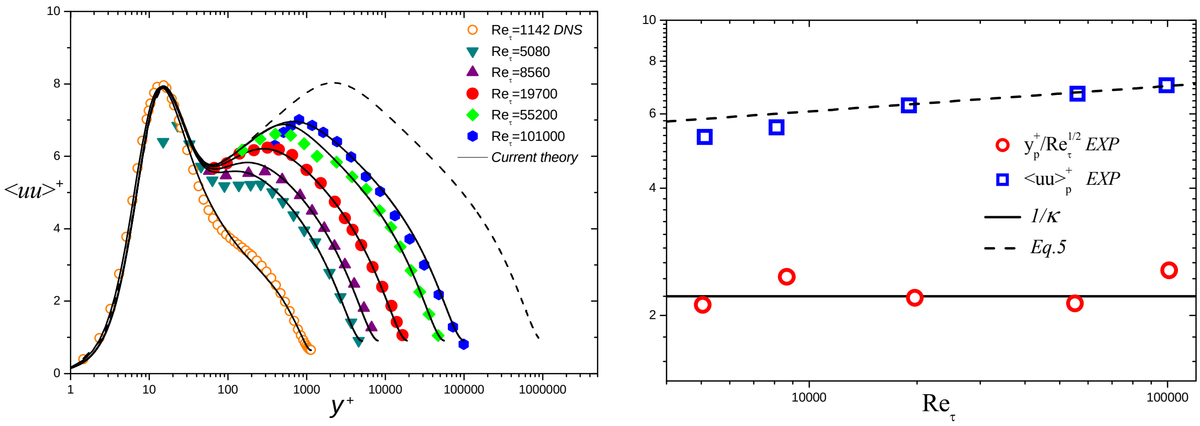

Fig.1 - Left: MKP as a function of the normalize distance from wall. Numerical data (solid) from [6]; experimental (open) from [1]; and the lines from the present model (dash line for $\mathrm{Re}_\tau = 10^6$). Right: The measured second peak location (red circle) and its magnitude (blue square) compared with theoretical predictions, namely Eqs.4 and 5.

DISCUSSION ON ANOMALY SCALING

By equaling Eq.4 to 150, we obtain a critical *Re* around 4550, below which Eq.4 ceases to be valid, and the flow enters a new regime. This is the transition between the two regimes with and without scaling anomaly (i.e. the second peak). A simple explanation is: the critical thickness $y^+ \approx 150$ is the boundary of the quasi-balance region between production and dissipation, hence the boundary of the bulk zone. For $Re<Re^{crit}$, the motion characterized by roll-up of vortex packets [7-8] is centered around the peak of W$^+$, which is less than 150, hence has no effective interaction with the bulk zone. When $Re>Re^{crit}$, these roll-up motions extend beyond 150, into the outer region; hence the inner near-wall motion effectively interacts with the bulk flow, leading to the growth of $\langle uu \rangle^+$ with a second peak and a scaling anomaly.

CONCLUSIONS

We have developed a quantitative model of MKP across the entire pipe for both DNS data at moderate Re and experimental data at high Re. A conjectured meso-layer is quantified and explained, together with the observed second peak in MKP at high Re. The present model of order length functions (ℓ_m, ℓ_u) provides a unified theory for describing both MVP and MKP. The procedure developed in this paper is general and should apply well to the other components of turbulent kinetic energy, and to other wall-bounded flows.

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