Cross-Independence Closure for Statistical Mechanics of Homogeneous Isotropic Turbulence

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Statistical mechanics of fluid turbulence has been formulated by Tatsumi [1] in terms of the closed equations for the multi-point velocity distributions under the cross-independence closure hypothesis proposed by Tatsumi [2]. In the present paper, this approach is applied to *homogeneous isotropic turbulence* and statistics of this turbulence is worked out.

Governing Equations for Homogeneous Isotropic Turbulence

For homogeneous isotropic turbulence, the minimum deterministic set of equations is provided by the closed equations for the *one-point velocity distribution* f,the *two-point velocity-sum distribution* g_+ and the *velocity-difference distribution* g_-, which are defined as

$$f(\mathbf{v}_1, t) = \langle \delta(\mathbf{u}_1(\mathbf{x}_1, t) - \mathbf{v}_1) \rangle, \tag{1}$$

$$g_\pm(\mathbf{v}_\pm, \mathbf{r}, t) = \langle \delta(\mathbf{u}_\pm(\mathbf{x}_1, \mathbf{x}_2; t) - \mathbf{v}_\pm) \rangle, \tag{2}$$

where $\mathbf{u}_\pm(\mathbf{x}_1, \mathbf{x}_2; t) = (\pm\mathbf{u}_1(\mathbf{x}_1, t) + \mathbf{u}_2(\mathbf{x}_2, t))/2$ denote the velocity-sum and -difference , and $\langle\ \rangle$ the statistical average.

The closed equations for these velocity distributions are derived from the corresponding equations for general turbulence in Tatsumi [1].

Equation for the distribution $f(\mathbf{v}_1, t)$:

$$\left[\frac{\partial}{\partial t} + \alpha(t)\left|\frac{\partial}{\partial \mathbf{v}_1}\right|^2\right] f(\mathbf{v}_1, t) = 0, \tag{3}$$

$$\alpha(t) = \frac{1}{3}\varepsilon(t) = \frac{2}{3}\nu \lim_{|\mathbf{r}|\to 0}\left|\frac{\partial}{\partial \mathbf{r}}\right|^2 \int |\mathbf{v}_-|^2 g_-(\mathbf{v}_-, \mathbf{r}, t), \tag{4}$$

where $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$, and $\varepsilon(t) = 3\alpha(t)$ denotes the *energy-dissipation rate* of turbulence.

The expression (4) of the energy dissipation rate $\varepsilon(t)$ in terms of the velocity-difference distribution g_-, which is mostly contributed from small-scale turbulence, actually manifests the "fluctuation-dissipation theorem" of non-equilibrium statistical mechanics.

Equation for the distribution $g_+(\mathbf{v}_+, \mathbf{r}, t)$:

$$\left[\frac{\partial}{\partial t} + \nu\frac{\partial}{\partial \mathbf{v}_+}\cdot\mathbf{v}_+\left|\frac{\partial}{\partial \mathbf{r}}\right|^2 + \frac{1}{2}\alpha(t)\left|\frac{\partial}{\partial \mathbf{v}_+}\right|^2\right] g_+(\mathbf{v}_+, \mathbf{r}, t) = 0. \tag{5}$$

Equation for the distribution $g_-(\mathbf{v}_-, \mathbf{r}, t)$:

$$\left[\frac{\partial}{\partial t} + 2\mathbf{v}_-\cdot\frac{\partial}{\partial \mathbf{r}} + \nu\frac{\partial}{\partial \mathbf{v}_-}\cdot\mathbf{v}_-\left|\frac{\partial}{\partial \mathbf{r}}\right|^2 + \frac{1}{2}\alpha(t)\left|\frac{\partial}{\partial \mathbf{v}_-}\right|^2\right] g_-(\mathbf{v}_-, \mathbf{r}, t)$$
$$= -\frac{1}{2}\frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{v}_-}\left\{\beta(-\mathbf{v}_-, t) + \beta(\mathbf{v}_-, t)\right\} g_-(\mathbf{v}_-, \mathbf{r}, t), \tag{6}$$

where the definition of the β terms is deleted for space.

Now, let us solve the equations (3), (5) and (6) for the distributions f, g_+ and g_-.

First, the change of the energy-dissipation rate $\varepsilon(t)$ in time is determined by dimensional analysis as $\varepsilon(t) = \varepsilon_0 (t/t_0)^{-2}$. Then, the decay of the energy $E(t) = \left\langle \left|\mathbf{u}_1(\mathbf{x}_1, t)\right|^2 \right\rangle /2$ is derived from the relation, $\varepsilon(t) = -dE(t)/dt$, as

$$E(t) = E_0 (t/t_0)^{-1}, \ .E_0 = \varepsilon_0 t_0. \tag{7}$$

Then, the solutions satisfying the similarity compatible to the energy decay law (7) are considered.

One-point velocity distribution

The one-point velocity distribution f is obtained from Eq.(3) as the inertial normal distribution (N1):

$$f(\mathbf{v}, t) = \left(\frac{t}{4\pi\alpha_0}\right)^{3/2} \exp\left[-\frac{|\mathbf{v}|^2 t}{4\alpha_0}\right], \quad \alpha(t) = \alpha_0 (t/t_0)^{-2}. \tag{8}$$

Two-point cross-velocity distributions (Global space)

In the global space, the $\mathbf{r}$-dependent terms in Eqs.(5) and (6) vanish, so that the equations become identical to Eq.(3) except for that the parameter $\alpha(t)$ is replaced by $\alpha(t)/2$. Thus the distributions $g_\pm$ are also expressed as another inertial-normal distribution (N2):

$$g_\pm(\mathbf{v}_\pm, t) = \left(\frac{t}{2\pi\alpha_0}\right)^{3/2} \exp\left[-\frac{|\mathbf{v}_\pm|^2 t}{2\alpha_0}\right]. \tag{9}$$

Thus, the distributions f and $g_\pm$ satisfy the convolution relation in the global space.

Two-point cross-velocity distributions (Local range)

The two-point distributions $g_\pm$ must satisfy different coincidence conditions, $g_+ \to f$, $g_- \to \delta$ for $|\mathbf{r}| \to 0$ and such conditions impose discontinuous changes to the distributions at $|\mathbf{r}| = 0$. In reality, however, such changes take place continuously in the local range which is expressed in terms of the local scale normalized by Kolmogorov's length $\eta = (\nu^3/\varepsilon)^{1/4}$.

Equations for the velocity distributions $g_\pm^*$ in the local range (symbolized by $*$) are derived from the corresponding equations in Tatsumi [1] and characterized by the *local energy-dissipation rates*,

$$\alpha_\pm^*(r^*, t^*) = \frac{1}{3}\varepsilon_\pm^*(r^*, t^*) = \frac{2}{3}\lim_{|\mathbf{r}^{*\prime}|\to 0}\left|\frac{\partial}{\partial \mathbf{r}^{*\prime}}\right|^2 \int \left|\mathbf{v}_{\pm-}^{*\prime}\right|^2 g_{\pm-}^{*\prime}\left(\mathbf{v}_{\pm-}^{*\prime}; \mathbf{r}^*, \mathbf{r}^{*\prime}; t^*\right) d\mathbf{v}_{\pm-}^{*\prime}. \tag{10}$$

The *local velocity-sum distribution* g_+^* is obtained by solving one of these equations as a local quasi-normal distribution associated with the parameter $\alpha_+^*(r^*, t^*) = \frac{1}{3}\varepsilon_+^*(r^*, t^*)$. The *local velocity-difference distribution* g_-^* is obtained by solving another one of these equations as a local quasi-singular distribution associated with the parameter $\alpha_-^*(r^*, t^*) = \frac{1}{3}\varepsilon_-^*(r^*, t^*)$.

References

[1] Tatsumi, T. 2011. Cross-independence closure for statistical mechanics of fluid mechanics. *J. Fluid Mech.* 670, 365-403.

[2] Tatsumi, T. 2001. Mathematical physics of turbulence. In Kambe et al eds. *Geometry and Statistics of Turbulence*, Kluwer Academic Publishers, Dordrecht, pp.3-12.