TOPOLOGICAL CHAOS AND MIXING IN FLUIDS STIRRED BY ALMOST-CYCLIC SETS

Mark A. Stremler[a)], Pradeep Rao, Shane D. Ross
Department of Engineering Science & Mechanics, Virginia Tech, Blacksburg, Virginia 24061, USA

Summary In certain (2+1)-dimensional fluid systems, the braiding of periodic orbits provides a framework for analyzing chaos in the system through application of the Thurston-Nielsen classification theorem (TNCT). Periodic orbits generated by the dynamics can behave as physical obstructions that 'stir' the surrounding fluid and serve as the basis for this topological analysis. Even in the absence of periodic orbits, 'almost-cyclic regions' can act as (leaky) obstructions that stir the fluid, facilitating application of the TNCT. This work merges topological and set-oriented approaches to analyzing fluid stirring as a dynamical system, using the strengths of each approach to enable predictive analysis in a broad class of fluid systems.

Mixing in fluid systems is an important phenomenon in a multitude of natural and industrial processes that cover a vast range of spatial and temporal scales. In many cases, the large-scale kinematics of the fluid motion plays a dominant role in the mixing process. This dependence on the large-scale flow kinematics is most often associated with mixing in laminar flows, but this can also be true in turbulent flows. For example, the spread of oil in the Gulf of Mexico following the Deepwater Horizon spill can be correlated with the motions of large-scale flow structures.

Significant understanding about the complexities of transport due to flow kinematics can be obtained by applying analysis tools from dynamical systems theory. The underlying concept is the fact that particles moving with a flow can exhibit chaotic trajectories even when the flow field is relatively simple. This phenomenon, known as chaotic advection, produces exponential stretching and folding of material lines and surfaces in the flow. The rapid lengthening and layering of interfaces due to chaotic advection is often augmented by molecular diffusion and leads to effective mixing. Identifying and understanding the mechanisms that produce or inhibit chaotic advection allow for systems that are "designed for chaos". Furthermore, identifying the dynamical structure of chaotic transport enables predictions of when and where mixing does, or does not, occur.

One relatively recent approach to analyzing chaos in a fluid system is the identification of 'topological chaos', or chaos due to topology of internal boundary motions such as moving rods [1]. This approach is based on application of the Thurston-Nielsen classification theorem to either two-dimensional, time-dependent flows or certain symmetric three-dimensional flows. 'Ghost rods', or periodic orbits generated by the dynamics, can also behave as physical obstructions that 'stir' the surrounding fluid, providing a basis for the topological analysis in place of physical boundaries [4]. With these topological approaches it is possible to observe the motion of a small number of objects, be they physical obstructions or ghost rods, and from the topology of their trajectories infer what possible, but otherwise unobserved, motions are present in the system.

The analysis of dynamical systems using the TNCT has depended primarily on the presence and identification of exactly periodic orbits [4]. The analysis of ghost rod topology in a fluid has been extended to aperiodic orbits [5], relaxing a substantial restriction in the application of the TNCT. However, this approach introduces the complexity of needing to identify appropriate aperiodic orbits, as a random selection of trajectories generally leads to a poor estimate of the overall system behavior [5].

In an alternative approach to analyzing stirring and mixing in a fluid, the domain (or phase space) is decomposed into subsets such that a typical trajectory has a very small probability of moving between subsets in a short time. In the examples we discuss there are two such subsets, or almost-invariant sets (AIS), one of which is often a disconnected set of multiple components that map one to another; these components are the almost-cyclic sets (ACS) of interest. The other subset is its phase space complement. One approach to analyzing stirring and mixing with this dynamical systems tool is to compute the eigenspectrum of the discretized Perron-Frobenius transfer operator, which provides insight into the various AIS present in the system [3]. The eigenvectors give the locations of the various AIS in the system, and the corresponding eigenvalues give measures of the invariance, or 'leakiness', of the AIS. The degree of mixing can be correlated with the invariance of the structure in the system. This approach is closely related to the 'mapping matrix' [6] and 'strange eigenmode' [2] methods for analyzing fluid mixing.

This talk explores a complementary perspective that tracks the time-dependent trajectories of the AIS components, or ACS, which can be viewed as individual physical obstructions that 'stir' the surrounding fluid. The space-time trajectories of the ACS are periodic, even though in general all particle trajectories are likely to 'leak' from these sets if given enough time. Sufficiently complex relative motions of the ACS lead to topological chaos that can be quantified using the TNCT. This approach combines the topological and set-oriented approaches, making it possible to investigate and quantify mixing in complex flows using limited information about the relative motion of a few coherent regions.

By way of example, we consider time-dependent flow in the two-dimensional domain $M = \{(x, y) : 0 \leq x \leq 2a, -b \leq$

a) Corresponding author. E-mail: stremler@vt.edu

$y \leq b\}$, shown in Fig. 1(a) for the Stokes' flow case. The flow is driven by a tangential velocity along the bottom boundary

$$V(x,t) = U_1 \sin(\pi x/2a + \phi) + U_2 \sin(\pi x/a + 2\phi),$$

$$\phi(t) = \begin{cases} \pi & \text{for } t \in [n\tau_f, (n+1)\tau_f/2), \\ 0 & \text{for } t \in [(n+1)\tau_f/2, (n+1)\tau_f), \end{cases}$$

where n is an integer, and a tangential velocity along the top boundary given by $-V(x,t)$, so that the flow is driven symmetrically about $y = 0$. We choose this example flow for three reasons: (i) in the limit of Stokes' flow, there is a closed-form mathematical representation for the velocity field; (ii) at finite Reynolds number, a spatially-continuous velocity field can be determined computationally using a spectral method; and (iii) this flow pattern is closely related to a class of experimentally generated counter-rotating flows.

Consider first the Stokes' flow case. For the parameter values we have chosen to consider here, taking $\tau_f = 1$ produces three period-three orbits in the flow that move about each other on space-time trajectories that can be represented by the braid on three strands shown in Fig. 1(f). The set-oriented approach identifies three ACS that surround these periodic orbits, as shown in Fig. 1(b), and the space-time trajectories of these ACS produce the identical braid. According to the TNCT, this braid is of *pseudo-Anosov* (pA) type. In the ideal pA case, material lines throughout the domain grow as λ_{TN}^p, where p is the (integer) period of the braid. This chaotic behavior is preserved under topological changes, so a subdomain of the fluid is subjected to exponential stretching at a rate that is at least λ_{TN}. Thus, the TNCT provides a lower bound on the topological entropy of the flow, $h = \ln(\lambda)$, where $\lambda \geq \lambda_{\text{TN}}$ is the maximum line-stretching exponent over all possible initial material lines. The size of the relevant subdomain is not predicted by the TNCT, but experimental results indicate that this region is typically on the scale of the orbit motions [1].

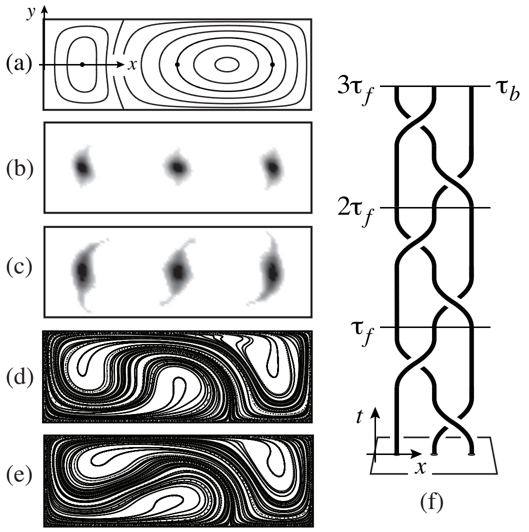

Figure 1. (a) Streamlines in the model lid-driven cavity flow in the limit of Stokes' flow during the first half-period. For the proper parameters, the circles mark periodic orbits that generate the braid in panel (f). (b–c) AIS/ACS in the limit of Stokes' flow for (b) the special case $\tau_f = 1$ and (c) a perturbed case with $\tau_f < 1$. (d–e) Patterns formed by an initially concentrated collection of passive particles after four periods of the flow for (d) Re $= 1$ and (e) Re $= 10$. (f) Braid generated by period motion of orbits and/or ACS in the flow.

If the boundary conditions are switched more rapidly, so that τ_f is slightly less than one, then even in the Stokes' flow case there are no longer any periodic orbits on which to base the analysis of the TNCT. However, there still exist ACS in the system that produce the same braiding motion, as illustrated in Fig. 1(c). Despite being only *almost* invariant, they obstruct the flow sufficiently to produce exponential stretching at a rate predicted by the TNCT. In fact, we have found that the leakiness of these 'stirring rods' leads to *improved* mixing, since they continually exchange fluid with their surroundings.

The presence of ACS as stirring rods also persists in this lid-driven cavity flow as the Reynolds number is increased from zero. Fig. 1(d–e) shows the influence of the ACS on a stirred concentration of passive particles for small, but non-zero, Reynolds number. Our analysis shows that as Re is increased beyond 10 the braiding structure is disrupted by inertial effects. Interestingly, and somewhat counterintuitively, this increase in Reynolds number beyond 10 is accompanied by a decrease in mixing efficiency.

We will discuss the relationships between the braiding of almost-cyclic sets (ACS), the predictions of the TNCT, and mixing efficiency. Almost-invariant sets, and the related 'strange eigenmode', have been observed in a variety of natural and engineering flows. Extending this analysis approach by considering the topology of the trajectories traced out by their disjoint components provides an important step in applying the power of the TNCT to a variety of practical flows.

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