CONCENTRATION INSTABILITY OF SEDIMENTING SPHERES IN A SECOND-ORDER FLUID

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Summary The sedimentation of spherical particles in non-Newtonian fluids is often characterized by the growth of density fluctuations. Using a linear stability analysis, we explain this effect in the case of a second-order fluid as a result of the nonlinear coupling between the settling motion of the particles and the mean-field flow driven by random density fluctuations, which causes particles to migrate towards the denser regions. We further characterize this instability using weakly nonlinear particle simulations based on a point-particle model.

INTRODUCTION

Experiments on the sedimentation of suspensions of spherical particles in highly viscous non-Newtonian fluids suggest that these systems develop clusters and large-scale density fluctuations as a result of hydrodynamic interactions between particles [1, 2, 3]. Such an effect does not occur in Newtonian fluids, where the suspensions remain homogeneous. However, other types of sedimenting suspensions are known to develop large-scale inhomogeneities: suspensions of non-spherical or non-rigid particles in Newtonian fluids are subject to concentration instabilities as a result of hydrodynamic interactions, also leading to the formation of large-scale concentrated structures [4, 5, 6]. These instabilities can be explained as a consequence of a coupling between the anisotropic mobility of the particles and the macroscale flow field generated by the density fluctuations in the suspension [4, 6]: density fluctuations create a disturbance flow which causes the particles to orient or deform in such a way that they migrate towards the regions of higher concentration. While this mechanism does not apply to isotropic particles in Newtonian fluids, a coupling between settling motion and local fluid flow is possible even for rigid spheres in non-Newtonian fluids owing to nonlinearities, suggesting that an instability similar to that for anisotropic particles may also exist in that case. In the present work, we elucidate this coupling and its consequences for suspension stability in the case of a second-order fluid.

THEORY

Mobility calculation

We first analyze the motion of a single isolated sphere of radius a sedimenting in a linear flow of the form $\mathbf{V}(\mathbf{x}) = \mathbf{V}_0 + \mathbf{x}\cdot\mathsf{A}$ where A is a constant linear gradient. The dynamics are governed by the mass and momentum conservation equations: $\nabla\cdot\mathbf{u} = 0$, $\nabla\cdot\boldsymbol{\sigma} = \mathbf{0}$. For a second-order fluid, the stress tensor is expressed as $\boldsymbol{\sigma} = -p\mathsf{I} + 2\eta_0\mathsf{E} + \boldsymbol{\Sigma}(\mathbf{u})$, where $\mathsf{E} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$ is the rate-of-strain tensor, and $\boldsymbol{\Sigma}(\mathbf{u})$ is the non-Newtonian stress contribution given by

$$\boldsymbol{\Sigma}(\mathbf{u}) = -\Psi_1[\mathbf{u}\cdot\nabla\mathsf{E} - \nabla\mathbf{u}^T\cdot\mathsf{E} - \mathsf{E}\cdot\nabla\mathbf{u}] + 4\Psi_2\mathsf{E}\cdot\mathsf{E} \tag{1}$$

in terms of the first and second normal stress differences Ψ_1 and Ψ_2. These equations have to be solved subject to the no-slip boundary condition on the surface of the particle, which moves with linear velocity $\mathbf{U}_p$ and angular velocity $\boldsymbol{\Omega}_p$: $\mathbf{u}(\mathbf{x}) = \mathbf{U}_p + \boldsymbol{\Omega}_p\times\mathbf{x}$, and to a force and torque balance on the particle, where the only external force is the gravitational force $\mathbf{F}$. We introduce the Deborah number (or dimensionless flow strength) $\mathrm{De} = \Psi_1\dot{\gamma}/\eta_0$ (where the characteristic shear rate is $\dot{\gamma} = (\mathsf{A} : \mathsf{A})^{1/2}$) and the ratio of the normal stress differences $\lambda = -4\Psi_2/\Psi_1$, and we note that the second-order fluid model is valid for $\mathrm{De} \ll 1$. In this limit, it is possible to solve for the linear and angular velocities $\mathbf{U}_p$ and $\boldsymbol{\Omega}_p$ as regular expansions in powers of De by application of the Lorentz reciprocal theorem for Stokes flow. All calculations done, we find the following expression for the linear velocity:

$$\mathbf{U}_p = \mathsf{M}_1(\mathsf{A})\cdot\mathbf{V}_0 + \mathsf{M}_2(\mathsf{A})\cdot\mathbf{F} + O(\mathrm{De}^2), \tag{2}$$

where the two tensors M_1 and M_2 are given by:

$$\mathsf{M}_1(\mathsf{A}) = \mathsf{I} + \frac{\Psi_1}{35\eta_0}(\mathsf{A} + \mathsf{A}^T), \quad \mathsf{M}_2(\mathsf{A}) = M_0\left\{\mathsf{I} + \frac{\Psi_1}{\eta_0}\left[f_1(\lambda)\mathsf{A} + f_2(\lambda)\mathsf{A}^T\right]\right\}, \tag{3}$$

with $M_0 = 1/6\pi\eta_0 a$, $f_1(\lambda) = \lambda/8 + 5/64$, and $f_2(\lambda) = \lambda/8 - 11/64$. In particular, we see that depending on the nature of the local velocity gradient $\mathsf{A} = \nabla\mathbf{V}$, the settling velocity of the sphere may have non-zero components in directions perpendicular to gravity.

Linear stability analysis

Using the result of equation (2), we can now perform a linear stability analysis for the evolution of the concentration field $c(\mathbf{x}, t)$ in a suspensions of identical spheres settling in a second-order fluid. By conservation of particles, the concentration c satisfies:

$$\partial_t c + \nabla\cdot(\dot{\mathbf{x}}c) - \nabla\cdot(\mathsf{D}\cdot\nabla c) = 0, \tag{4}$$

where we allow for diffusion with constant diffusivity tensor D. In equation (4), $\dot{\mathbf{x}}$ denotes the velocity a particle at position $\mathbf{x}$ in the fluid, and is obtained by application of equation (2) as $\dot{\mathbf{x}} = \mathsf{M}_1[\nabla\mathbf{u}(\mathbf{x})]\cdot\mathbf{u}(\mathbf{x}) + \mathsf{M}_2[\nabla\mathbf{u}(\mathbf{x})]\cdot\mathbf{F}$. The mean-field fluid velocity is driven by the gravitational force acting on the density fluctuations in the fluid, and satisfies

$$\nabla\cdot\mathbf{u} = 0, \quad -\eta_0\nabla^2\mathbf{u} + \nabla p = \nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{u}) + \mathbf{F}c(\mathbf{x}). \tag{5}$$

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A linear stability analysis can be performed on these equations, taking the base state to be uniform in space. Assuming a plane-wave perturbation of the form $c(\mathbf{x}, t) = c_0 + \tilde{c}(\mathbf{k}) \exp(\mathrm{i}\mathbf{x} \cdot \mathbf{k} + \sigma t)$, the following expression can be found for the growth rate σ:

$$\sigma = \frac{M_0 n \Psi_1 f_1(\lambda) F^2}{\eta_0^2} \sin^2 \Theta - \mathbf{k} \cdot \mathbf{D} \cdot \mathbf{k} - \mathrm{i} M_0 F k \cos \Theta, \tag{6}$$

where Θ is the angle between $\mathbf{F}$ and $\mathbf{k}$. Noting that $\Psi_1 \geq 0$ and $f_1(\lambda) \geq 0$ for most polymeric fluids, we find that the real part of σ is positive at low wavenumbers for wavevectors $\mathbf{k}$ that have a non-zero component in a direction perpendicular to gravity ($\sin \Theta \neq 0$). At high wavenumbers, particle diffusion damps this growth rate and stabilizes the suspension. The growth rate is plotted as a function of the wavenumber for various wave angles in figure 1, in which dimensionless variables are used. The mechanism for the instability is qualitatively similar to that described by Koch & Shaqfeh [4] for spheroids. Specifically, a weak concentration fluctuation in the suspension (e.g. due to the initial random mixing) will create a disturbance flow, which points downwards in the high-density regions and upwards in the low-density regions. When a sphere sediments in this flow, the nonlinear coupling of its settling motion with the local flow field owing to viscoelastic effects causes it to migrate towards the denser regions, according to the mobility calculation described above. This migration has the effect of reinforcing the initial concentration fluctuation, thereby leading to the growth of inhomogeneities.

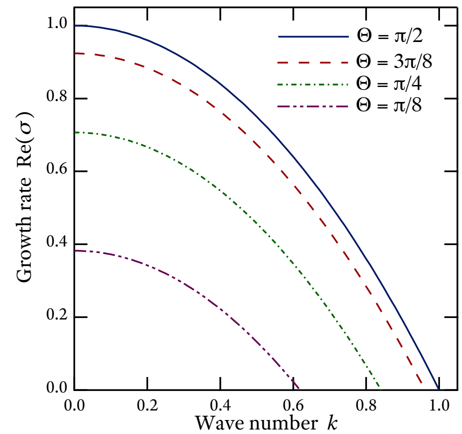

Figure 1. Growth rate, defined as the real part of σ in equation (6) as a function of wave number k, for different wave angles Θ.

NUMERICAL SIMULATIONS

To confirm the existence of the instability and to study the dynamics and pattern formation in the nonlinear regime, we also perform direct particle simulations in the limit of weak viscoelasticity (low Deborah number). The simulation method is based on a point-particle model similar to that previously used by Saintillan *et al.* [6], and uses an iterative spectral solution of the momentum equation to capture viscoelastic effects to $O(\mathrm{De})$ in both particle and fluid motions. A slip boundary condition is enforced on the container walls, as in the previous work of Bergougnoux *et al.* [7]. Typical results from these simulations are illustrated in figure 2, and confirm the existence of an instability. When $\mathrm{De} > 0$, we find that particles aggregate into dense large-scale clusters, which tend to organize into vertical streamers, in qualitative agreement with previous experiments [3]. These clusters, which settle faster than isolated particles, have a strong impact on the sedimentation statistics, and result in a strong enhancement of both the mean settling speed and velocity fluctuations, see figure 2(c)-(d). A strong stable density stratification is also observed to develop.

References

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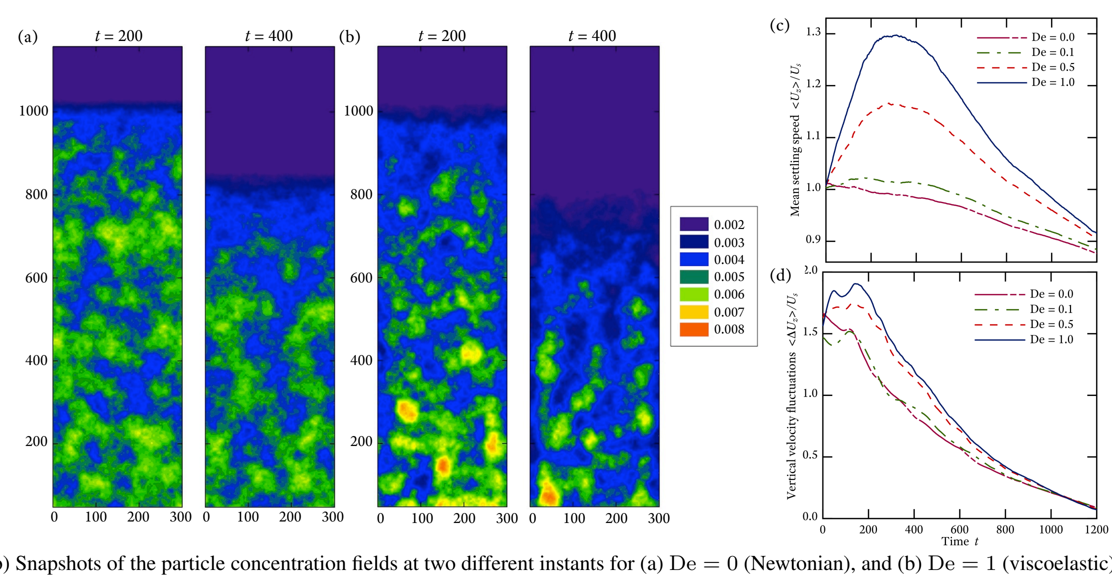

Figure 2. (a)-(b) Snapshots of the particle concentration fields at two different instants for (a) $\mathrm{De} = 0$ (Newtonian), and (b) $\mathrm{De} = 1$ (viscoelastic), in a simulation of $\sim$130,000 particles with volume fraction $\phi = 0.5\%$. (c) Mean velocity, and (d) velocity fluctuations as functions of time at different Deborah numbers.