MICROPOLAR THEORY AND ITS APPLICATION TO NEMATIC LIQUID CRYSTAL

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Summary Micropolar theory, developed by Eringen [1-3], is the extension of the classical field theory to microscopic space and time scales. It envisions a material body as a continuous collection of finite-size particles; each is capable of moving and rotating. The purpose to go beyond the classical continuum theory is to take into account the microstructure of the material body in question while still keeping the advantages of continuum theory intact. Current developments in microcontinuum theory may be found in [4-5].

KINEMATICS AND BALANCE LAWS

Geometrically, a deformable particle $\mathbf{P}$ is characterized by its centroid C located at $\mathbf{X}$ and vector $\mathbf{\Xi}$ relative to $\mathbf{X}$ of a generic point within the particle, at time $t = 0$. The material is called Microcontinuum, if the deformation carrying $\mathbf{P}(\mathbf{X}, \mathbf{\Xi}, t)$ to $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}, t)$ in the deformed state at time t (cf Fig. 1) can be expressed as

$$x_k = x_k(\mathbf{X}, t) \qquad , \qquad \xi_k = \chi_{kK}(\mathbf{X}, t)\Xi_K \tag{1}$$

where the microrotation χ_{kK} can be expressed as $\chi_{kK} = \lambda_{kl}\delta_{lK}$ with

$$\lambda_{kl} = \cos\phi\delta_{kl} - \sin\phi\epsilon_{klm}n_m + (1-\cos\phi)n_k n_l, \qquad \phi \equiv \{\phi_k\phi_k\}^{1/2}, \qquad n_k = \phi_k/\phi \tag{2}$$

The balance laws of micropolar theory are obtained as

$$\dot{\rho} + \rho v_{k,k} = 0, \tag{3}$$

$$\frac{dj_{kl}}{dt} + (\epsilon_{kmn}j_{lm} + \epsilon_{lmn}j_{km})\omega_n = 0, \tag{4}$$

$$t_{kl,k} + \rho f_l = \rho\dot{v}_l, \tag{5}$$

$$m_{kl,k} + \epsilon_{lmn}t_{mn} + \rho l_l = \rho\sigma_l, \tag{6}$$

$$\rho\dot{e} = m_{kl}\omega_{l,k} + t_{kl}(v_{l,k} + \epsilon_{lkm}\omega_m) - q_{k,k} + \rho h, \tag{7}$$

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + m_{kl}\omega_{l,k} + t_{kl}(v_{l,k} + \epsilon_{lkm}\omega_m) - q_k\theta_{,k}/\theta \geq 0 \tag{8}$$

where $\mathbf{v}$, $\boldsymbol{\omega}$, $\mathbf{t}$, $\mathbf{m}$, e, $\mathbf{q}$, h, η, θ, and $\psi \equiv e - \eta\theta$ are the velocity, angular velocity, Cauchy stress, moment stress, internal energy density, heat flux, heat source, entropy density, absolute temperature, and the Helmholtz's free energy density, respectively; and the spin inertia is defined as $\sigma_k \equiv j_{kl}\dot{\omega}_l + \epsilon_{klm}j_{mn}\omega_l\omega_n$.

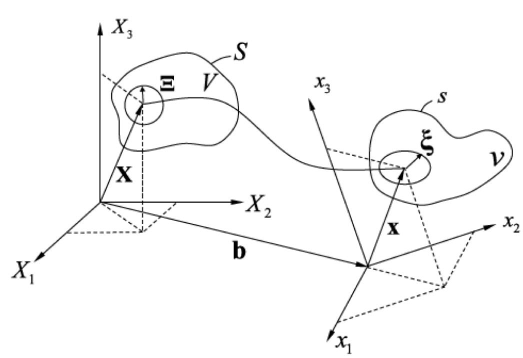

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Figure 1. Macro- and micromotions of a material particle.

CONSTITUTIVE EQUATIONS OF NEMATIC LIQUID CRYSTAL

Definition: Nematic liquid crystal, a subclass of micropolar viscoelastic solid, is a material body that takes any state leaving density unchanged and gradient of microrotation vanished as reference state.

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Following this definition rigorously, the constitutive equations for nematic liquid crystal are obtained as

$$\begin{aligned} t_{kl} &= -p\delta_{kl} + j^{-1}\{\bar{a}_{kjmn}\dot{c}_{mn} + \bar{d}_{kjmn}\dot{\gamma}_{mn}\}\lambda_{lj} \\ m_{kl} &= j^{-1}\{\bar{B}_{jkmn}\alpha_{mn} + \bar{b}_{jkmn}\dot{\gamma}_{mn} + \bar{d}_{mnjk}\dot{c}_{mn}\}\lambda_{lj} \end{aligned} \tag{9}$$

where

$$\alpha_{kl} \equiv \frac{1}{2}\epsilon_{kmn}\lambda_{rm,l}\lambda_{rn}, \dot{c}_{kl} \equiv (v_{m,k} + \epsilon_{mkn}\omega_n)\lambda_{ml}, \dot{\gamma}_{kl} \equiv \omega_{m,l}\lambda_{mk}, j \equiv det(x_{k,K}) \tag{10}$$

$$\{\bar{B}, \bar{a}, \bar{b}, \bar{d}\}_{kjmn} \equiv \{B, a, b, d\}_{KJMN}\delta_{kK}\delta_{jJ}\delta_{mM}\delta_{nN} \tag{11}$$

and $p = p(\rho)$ is the thermodynamic pressure; $\{B, a, b, d\}_{KJMN}$ are the material properties; $\delta_{kK} \equiv \cos\theta_{(k,K)} = \mathbf{e_k} \cdot \mathbf{e_K} = \mathbf{e_K} \cdot \mathbf{e_k} = \delta_{\mathbf{Kk}}$ is the *shifter* which is a matrix made of the direction cosines between two coordinate systems: the Eulerian coordnates and the Lagrangian coordinates – characterized by the most current orientation of the nematic liquid crystal (averaged over the whole system). It is seen that, from eq.(9), if the rate terms are vanishing ($\mathbf{c} = \boldsymbol{\gamma} = \mathbf{0}$), then the Cauchy stress becomes thermodynamic pressure which is a function of density, and the moment stress reduces to zero if the gradient of microrotation vanishes, i.e., $\nabla\boldsymbol{\lambda} = \mathbf{0}$; it means the nematic liquid crystal prefers its molecules to remain parallel to each other. The constitutive equation indeed reflects the definition of the nematic liquid crystal. On the other hand, the material properties indicate anisotropy (cf eq.(11)) which is the characteristic of solid material(compared to flow).

FLOW PROBLEM

Here we investigate the Couette flow problem of nematic liquid crystal. This problem is a typical one in the observation of effects due to shear viscosity of fluids. We take the well known physical phenomenon, orientation of nematic liquid crystal being parallel to the static magnetic field, as the starting point. Our investigations of the Couette flow indicate that if the shear flow in the direction of the orientation is a solution to the problem (cf Fig.2a), then it is impossible to have a shear flow perpendicular to the orientation unless a magnetic field or boundary orienting forces exists to maintain the orientation (cf Fig.2b).

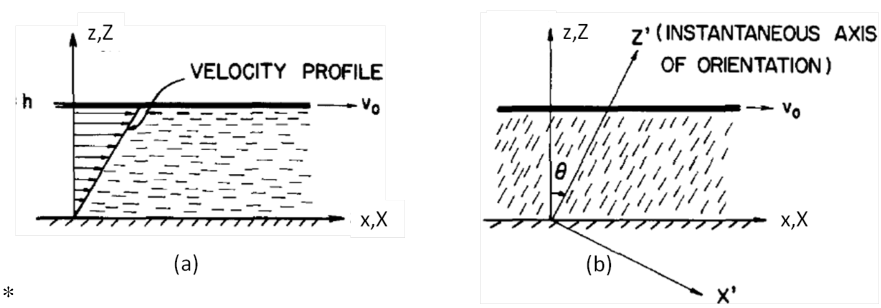

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Figure 2. (a) Couette flow in the direction of orientation; (b) Change of orientation due to shear flow (eventually $\theta \to 90°$).

CONCLUSIONS

In comparison with classical continuum theory, we note that the change of orientation due to magnetic field is the example showing that the additional degrees of freedom, the microrotation, must be added in treating liquid crystals. The flow problems solved here indicate the existence of coupling between velocity field and rotational field. Also, the characteristic of liquid crystal, material anisotropy, that is usually associated with solid, plays a vital role[6].

References

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