# LAGRANGIAN TORI INDUCED BY PRECESSING BENT RODS IN VISCOUS DOMINATED FLOWS.

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*Summary*
Motile cilia play a large role in fluid motion across the surface of ciliated tissue. In this paper, we approximate the cilium as a slender body rotating in a viscous fluid about one of its ends in contact with a horizontal no-slip plane. The chord connecting both ends of the bent rod sweeps out an upright cone. The bend of the rod, the cone angle (the angle between the chord and the vertical axis of rotation), and the orientation of the bent rod about its chord affect the flow patterns in a Stokes fluid. A slender body theory allows for an asymptotic solution of the Lagrangian trajectories and flow patterns caused by the precessing rod, which can be directly compared to experimental data. Altering the above parameters produces different toroidal flow structures. Using 3D stereographic imaging techniques, quantitative comparisons of epicyclic particle trajectories in short and long time are made against the model predictions.

## PROBLEM SETUP

We consider the evolution of Lagrangian markers in a viscous fluid flow generated by the motion of a rigid, thin uniformly curved rod in an arc of a circle whose chord sweeps an upright cone. Our study has both experimental and theoretical components. The geometry of the problem is sketched in Figure 1. The sketch is realized in our experimental setup within a plexiglass tank filled with either silicone oil or corn syrup, and a soft magnetic metal rod with one sharpened in contact with the tank's floor. The cubic tank has 30 cm sides, and the rod length is 1cm, and its diameter is $0.074$cm. The silicone oil and corn syrup typical kinematic viscosities are $12,500$ and $2,000$ cSt, respectively. A permanent magnet is rotated beneath the tank to provide external actuating through magnetic coupling with the rod. Micro-balloons of radius 30 $\mu$m are initially positioned in the fluid and are used as Lagrangian tracers. The rod and tracer locations are collected through stereographic techniques using two orthogonally placed cameras, calibrated using the Camera Calibration Toolbox [1]. The video frames record silhouettes of the rod and tracers which are then tracked through digital software. The rod rotation rates are very slow, with Reynolds number of $0.001$ and smaller.
The dynamic rod position and orientation are input into a slender body asymptotic fluid model based upon fundamental solutions of the Stokes equations, and their companion images to account for the no-slip boundary condition on the floor. The resulting analytic expression for the velocity field is then numerically integrated in time to compute particle trajectories, which are subsequently compared to the experimental data.

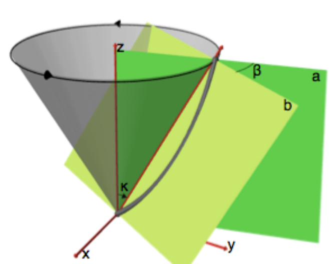


**Figure 1.** Sketch of the geometry of the precessing bent rod and its orientation

## RESULTS

Our previous work demonstrated that our slender body asymptotic theory provides quantitative, accurate agreements with the experiment for *straight* rods over the span of 10 rod precessions [2]. Interestingly, the Lagrangian trajectories for the straight rod configurations live nearly in horizontal planes, showing no net vertical transport. In contrast, the role of bend triggers large scale vertical transport organized along nested tori. This is demonstrated in Figure 2 which depicts the intersection of the particle trajectory with a vertical plane, generating a return map. As the orientation angles $(\beta, \kappa)$ vary, significant vertical transport is generated, particularly as the angle $\beta$ approaches $\pi/2$. In turn, as $\beta$ approaches $0$ or $\pi$, the

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rate of the vertical transport vanishes, but significantly, not its spatial extent. This reflects a broken symmetry induced by off-plane orientation parametrized by the angle $\beta$.

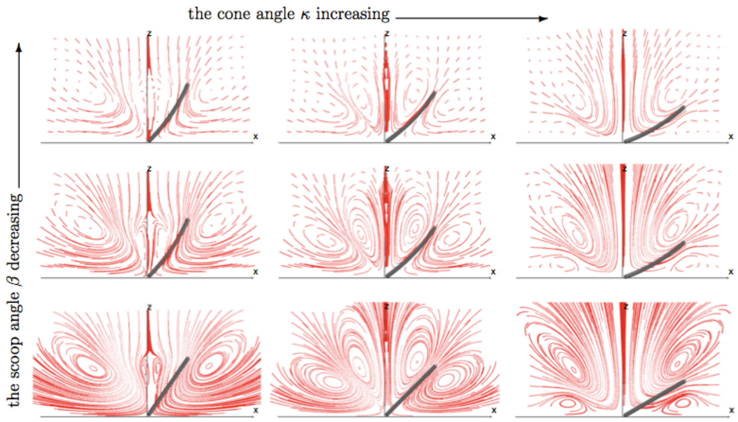


**Figure 2.** Theoretical return map for Lagrangian trajectories as orientation angles, $\beta$ and $\kappa$ are varied.

The corresponding three dimensional experimental data and theoretical comparison are depicted in Figure 3, with the experimental trajectory on the left, and the theoretical trajectory in the center. Shown on the right panel is a photo of the experimental torus as marked by a dye stream, with the inset showing a computed theoretical trajectory. Both the experimental and theoretical trajectories slowly wind the tori in a qualitatively similar fashion. We emphasize that the timescale to wind the complete torus in either case is exceptionally long (hundreds of rod rotations), and a completely accurate quantitative comparison on these long timescales is not feasible. We also emphasize that for the case of a straight rod (for which the Lagrangian trajectories do not wind tori) accurate agreement between model and experiment are obtained for moderate timescales [2], while in the present case involving a bent rod accuracy erodes after only a few rod rotations.

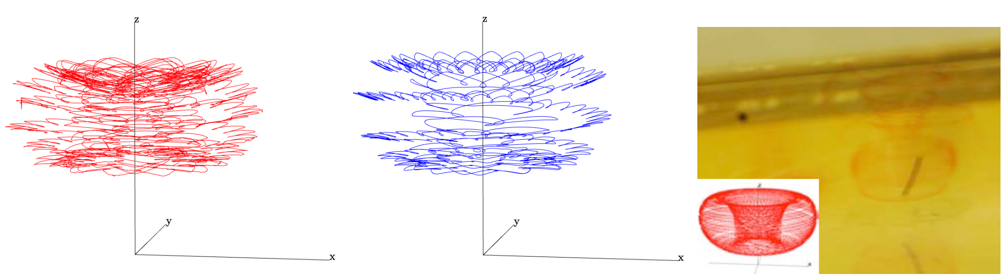

**Figure 3.** Three dimensional experimental (left), theoretical (center) Lagrangian trajectories, and photo of experimental torus with theoretically computed torus in the inset.

## CONCLUSIONS

Experiments were performed with bent rods precessing upright cones in viscous fluids in which long time Lagrangian trajectories were tracked and observed to slowly wind nested tori. Theoretical calculations based on slender body demonstrate the origin of these tori as a symmetry breaking induced by the orientation of the rod bend as it precesses through the fluid. Comparisons between the experimental and theoretical trajectories show qualitative geometric agreement, however, short time quantified agreement is lost on shorter timescales than the straight rod counterpart. We note that numerous corrections to the theory (including alternative slender body theories correcting for tip behavior, and moderate curvature) in the steady Stokes regime have not produced extended quantitative agreement, and future work will address improved short time corrections exploring non-zero Reynolds and Strouhal effects. **Acknowledgements:** The authors thanks NSF for supporting this work through NSF RTG DMS-0502266, NSF RTG DMS-0943851, and NSF DMS- 1009750.

## References


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