SETTLING OF A POROUS SPHERE IN STRATIFIED STOKES FLOWS

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Summary The effect of porosity and salt diffusion in the dynamics of a sphere settling under gravity in a salt-stratified fluid is studied analytically in viscosity dominated regimes. An explicit solution for the sphere's position in time is derived, under the assumption that settling time scales be much larger than diffusion time scales, as defined by the ambient salinity gradient, the sphere's diameter, and its velocity. Based on the explicit solution, a parametric study of the settling behaviors is performed and presented. Preliminary comparisons with experimental data show qualitative agreement with the results shown here.

INTRODUCTION

Marine snow is fundamental to marine carbon cycling both in its importance in carbon flux from the surface ocean and because it serves as hot spots for microbial activity. Understanding the function of marine snow in carbon biogeochemistry requires knowledge of the small-scale settling dynamics of marine snow particles in different physical environments. Marine snow particles are macroscopic aggregates composed of organic and inorganic material from the ocean [1], [2].

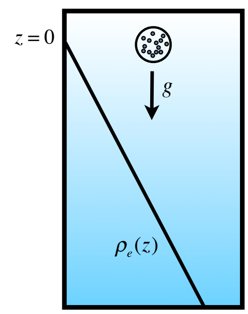

Figure 1. This schematic shows a porous sphere settling in an ambient fluid with a linear density gradient.

Settling of marine snow in physical environments is a complex problem. Most particles are extremely porous, allowing for diffusion of salt from the environment to change the density of the fluid in the pores, thus affecting the total density of the conglomerate. This flux of salt into particles can vary to various degrees depending on the nature of density gradients of the ambient fluid environment and the particle settling speeds, as well as the advection of fluid by the moving particles. This gives rise to different competing rates that play a role in this problem, such as the rate of diffusion and the rate of particle settling. The modeling aspects of previous work [3] have approached the problem numerically using an empirically derived drag law for a settling sphere in stratified fluids. In our approach, we enforce Archimedean force balance under Stokes flow conditions and present an exact formula for the sphere position as it settles in time. The model relies on the assumption that the time scale of diffusion is comparable or smaller than the settling time scale, as determined by the ambient salinity gradient and the settling speed of the particle. The ambient fluid is assumed to be a reservoir with a prescribed density depending only on depth, $\rho_e(z)$. (Effectively all of these assumptions ensure the Stokes flow approximation applies.) A sketch of the problem being analyzed here is given in Figure 1. Here, results are presented in the case of a linear density gradient in the ambient fluid, $\rho_e(z) = m\,z + \rho_e(0)$.

PROBLEM SETUP

The position of the sphere's center at time t, $z = Z(t)$, is determined by Stokes drag law,

$$6\pi\mu a \frac{dZ}{dt} = g\left(M_{\text{sphere}} - \rho_e(Z(t))\right)V\,, \tag{1}$$

where μ is the viscosity of the ambient fluid, V is the volume of the sphere with radius a, $V = \frac{4}{3}\pi a^3$, and $\rho_e(z)$ is the density of the ambient fluid when the particle is at depth z. The mass of the sphere is $M_{\text{sphere}} = (1-P)V\rho_{\text{solid}} + PM_{\text{fluid}}$, where P is a measure of the porosity, the proportion of the sphere which is fluid, ρ_{solid} is the density of the solid components of the sphere, and M_{fluid} is the time dependent mass of the fluid within the sphere. An assumption of constant temperature and pressure is made in this problem which allows salinity and density to be related linearly, $c = \alpha\rho + \beta$. Therefore, the mass of the saltwater within the sphere can be computed from the salinity of the water in the sphere, $c(\vec{x}, t)$, $M_{\text{fluid}} = (-\beta V + \int_V c(\vec{x}, t)\,dV)/\alpha$.

Porous diffusion

Using conservation of mass and Fick's law, the salinity within the porous sphere is governed by the heat equation, $c_t = \kappa\nabla^2 c$, where κ is the salt diffusivity. We assume spherical symmetry, and constant initial salinity, $c(r, 0) = c_o$. For the boundary condition, we take $c(a, t) = f(t)$. This function couples the diffusion problem to the sphere's motion, $f(t) = c_e(Z(t)) = \alpha\rho_e(Z(t)) + \beta = \alpha m Z(t) + \alpha\rho_e(0) + \beta$. This assumption of a spatially independent boundary condition is justifiable for length scales of (vertical) variations of salinity which are much larger than the sphere's diameter. The heat equation is solved analytically, e.g. by Laplace transform $L[\cdot](s)$, to give

$$L\left[\int_V c\,dV\right](s) = 4\pi a\left(L[f](s) - \frac{c_o}{s}\right)\left(a\sqrt{\frac{\kappa}{s}}\coth\left(a\sqrt{\frac{s}{\kappa}}\right) - \frac{\kappa}{s} + \frac{4}{3}\pi a^3\frac{c_o}{s}\right)\,. \tag{2}$$

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Sphere position

The Stokes drag law, equation (1), can be solved explicitly in the Laplace domain for the sphere position,

$$L[Z](s) = \frac{B}{A_1 s^2 + A_2 + A_3 s + A_4 \sqrt{s} \coth\left(a\sqrt{\frac{s}{\kappa}}\right)} \tag{3}$$

where $A_1 = 6\pi\mu a$, $A_2 = 4\pi P\, g\, a\, \kappa\, m$, $A_3 = Vmg$, $A_4 = -4\pi\sqrt{\kappa}\, P\, g\, a^2\, m$, and $B = Vg(1-P)(\rho_{\text{solid}} - \rho_e(0))$. This form of the solution assumes that the initial position of the sphere is at the origin, $Z(0) = 0$, and that the initial salinity within the sphere is equal to that of the ambient fluid at the same depth, $c_o = c_e(0)$.

Asymptotics

From this analytic solution the expected short and long time behavior of the sphere location can be observed. For large times, the sphere position approaches the depth at which the ambient fluid density is equal to the solid component density of the sphere. For small times, the sphere settles at a constant velocity equal to the settling Stokes velocity of a solid sphere with density equal to the initial density of the porous sphere. More refined information on short and long time functional dependence can be extracted using asymptotic Tauberian relations.

Parametric studies

In Figure 2, some preliminary parameter studies are presented. To solve for $Z(t)$, the inverse Laplace transform of equation (3) is computed numerically. Here, depth of the sphere versus time is shown for varying values of the porosity, P, and for varying slopes of the ambient linear density gradient, m. Physical values of the porosity are usually greater than

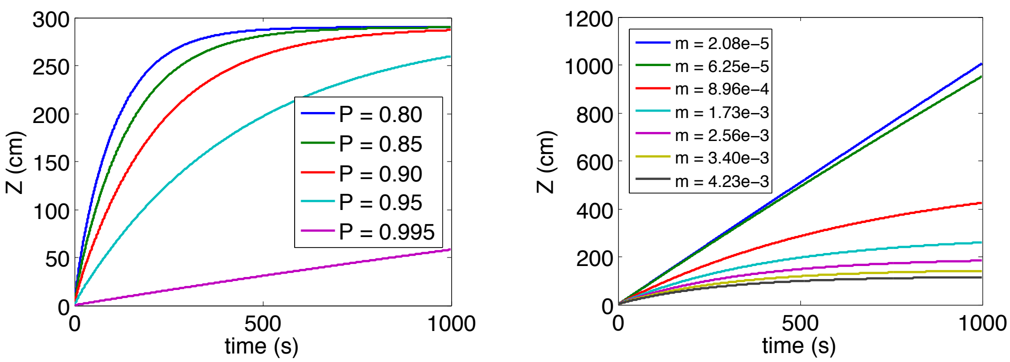

Figure 2. Parametric study varying the porosity, P, (left) and varying the slope of the ambient density gradient (right). For both studies, $a = 0.04355\,\text{cm}$, $\rho_{\text{solid}} = 1.5\,\text{g/cm}^3$, $\mu = 0.01002\,\text{g/cm/s}$, $\kappa = 1.5 \times 10^{-5}\,\text{cm}^2/\text{s}$, and $\rho_e(0) = 0.9985\,\text{g/cm}^3$. In the porosity study, $m = 0.00173\,\text{g/cm}^4$ and in the ambient density study, $P = 0.95$.

0.95. Notice that within the range of porosity shown here, some cases (lower porosity) lead to a large second derivative while other cases (lower porosity) do not. The slope of the ambient density gradient $m = 1.73 \times 10^{-3}\,\text{g/cm}^4$ is the one used in some preliminary experiments. The range of slopes presented here agrees with expected behavior: smaller density gradients lead to near constant settling speeds (and longer times to reach equilibrium depth), while steeper gradients cause more rapid reductions of the settling velocity in approaching the equilibrium depth. These studies easily allow prediction of the different behaviors that can be seen in experiments as certain parameters are varied.

CONCLUSIONS AND FUTURE WORK

We have presented a closed form solution for the problem of a porous sphere settling slowly in a linear density gradient ambient fluid. From this analytic solution, a few preliminary parametric studies were conducted. Our analytical approach can be extended to more general cases of stratification resulting in a nonlinear integral differential equation whose study will be left for future work. Current work is focusing on developing experimental verifications of our theoretical predictions.

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References

[1] Alldredge A. and Silver M.: Characteristics, Dynamics and Significance of Marine Snow. *Prog. Oceanog* **20**:41-82,1988.
[2] Turner J.: Zooplankton Fecal Pellets, Marine Snow and Sinking Phytoplankton Blooms. *Aquat Microb Ecol* **27**:57-102, 2002.
[3] Kindler K., Khalili A., and Stocker R.: Diffusion-limited Retention of Porous Particle at Density Interfaces. *PNAS* **105:51**:22163-22168, 2010.