STOKESLET INDUCED SLIP FLOWS IN A SPHERICAL CONTAINER

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Summary We discussed the Stokes flow generated due to a Stokeslet located inside a spherical container using Navier slip boundary conditions. For the axisymmetric flow generated due to a Stokeslet at the origin, we conjectured that the co-existence of a stagnation point and a point of zero vorticity is a necessary condition for the eddy patterns to occur. In the case of an axisymmetric off-centred Stokelet, the flow patterns remain the same due to the same set of points of zero vorticity for slip-parameter values greater than 0.5.

INTRODUCTION

Consider the Stokes flow generated due to a Stokeslet located at the origin, inside a spherical container of radius 1. The governing equations in a non-dimensional form are

$$\nabla^2\vec{\mathbf{q}} = \nabla p - \vec{\mathbf{F}}\delta(\vec{\mathbf{r}}), \tag{1}$$
$$\nabla.\vec{\mathbf{q}} = 0, \tag{2}$$

where $\vec{\mathbf{q}}$ is the velocity and p the pressure of the fluid, $\delta(\vec{\mathbf{r}})$ is the Dirac delta function and $\vec{\mathbf{F}}$ is the strength of the Stokeslet, which are all expressed in a non-dimensional form using a characteristic velocity U and a characteristic length a. The slip boundary conditions on $r = 1$ are given in a non-dimensional form as follows [1].

$$q_r(1,\theta,\phi) = 0, \tag{3}$$
$$q_\theta(1,\theta,\phi) = -\lambda T_{r\theta}(1,\theta,\phi), \tag{4}$$
$$q_\phi(1,\theta,\phi) = -\lambda T_{r\phi}(1,\theta,\phi), \tag{5}$$

where λ is the non-dimensional slip parameter, (q_r, q_θ, q_ϕ) is the non-dimensional velocity and $T_{r\theta}$, $T_{r\phi}$ are the corresponding tangential stresses given by $T_{r\theta} = \left(\frac{1}{r}\frac{\partial q_r}{\partial\theta} - \frac{q_\theta}{r} + \frac{\partial q_\theta}{\partial r}\right), T_{r\phi} = \left(\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} - \frac{q_\phi}{r} + \frac{\partial q_\phi}{\partial r}\right)$ in spherical polar co-ordinates (r,θ,ϕ). We use the following representation for the velocity and pressure of the fluid [2] as follows.

$$\vec{\mathbf{q}} = \nabla\times\nabla\times(\vec{\mathbf{r}}A) + \nabla\times(\vec{\mathbf{r}}B), \tag{6}$$
$$p = \frac{\partial}{\partial r}(r\nabla^2 A), \text{ where } \nabla^4 A = 0,\ \nabla^2 B = 0. \tag{7}$$

For $\lambda \neq 1$, the boundary conditions on $r = 1$, in terms of A and B, are $A = 0,\ \frac{\partial A}{\partial r} = -\lambda\frac{\partial^2 A}{\partial r^2},\ B = -\left(\frac{\lambda}{1-\lambda}\right)\frac{\partial B}{\partial r}$.
For $\lambda = 1$, the boundary conditions on $r = 1$ are $A = 0,\ \frac{\partial A}{\partial r} = -\frac{\partial^2 A}{\partial r^2},\ \frac{\partial B}{\partial r} = 0$.

AXISYMMETRIC CASE

When the flow is axisymmetric, the stream function $\psi(r,\theta)$ and the scalar functions A and B have the following relations: $A = A(r,\theta),\ B \equiv 0,\ \frac{\psi(r,\theta)}{r\sin\theta} = \frac{\partial A(r,\theta)}{\partial\theta}$. In terms of the stream function, the boundary conditions on $r = 1$ are

$$\psi(1,\theta) = 0,\ \frac{\partial\psi(1,\theta)}{\partial r} = -\left(\frac{\lambda}{1-2\lambda}\right)\frac{\partial^2\psi(1,\theta)}{\partial r^2}.$$

For $\lambda = \frac{1}{2}$ the boundary conditions are $\psi(1,\theta) = 0,\ \frac{\partial^2\psi(1,\theta)}{\partial r^2} = 0$. The second condition has a deeper implication that the vorticity on the boundary vanishes at $\lambda = \frac{1}{2}$, indicating flow '*separation*'. We studied the following examples.

1. **Stokeslet of strength** $(0,0,\frac{F}{8\pi})$ **located at** $(0,0,0)$ [3]: The perturbed flow generated inside $r = 1$ is

$$\psi(r,\theta) = \frac{F}{8\pi}\left[r - \frac{3(1+2\lambda)}{2(1+3\lambda)}r^2 + \frac{1}{2(1+3\lambda)}r^4\right]\sin^2\theta.$$

Using our exact solution for the Stokeslet at origin we have plotted the streamline patterns for (a) $\lambda = 0,00042$, (b) $\lambda = 0.0085$, (c) $\lambda = 0.03$, (d) $\lambda = 5$. The following table and the plots reveal the fact that the co-existence of

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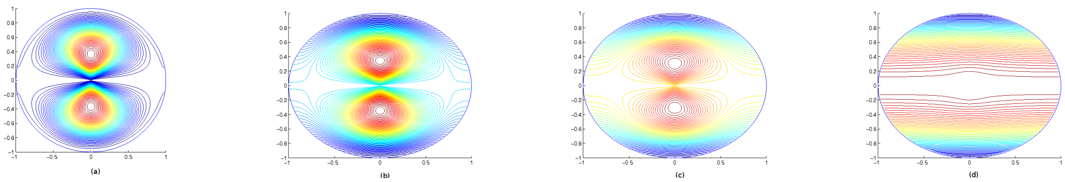
stagnation point (at $r = r_{\text{stag}}$) and the point of zero vorticity (at $r = r_\Omega$) in the flow domain is necessary for the existence of eddy patterns.

λ	0	0.001	0.1	0.5	5	10
r_{stag}	0.3660	0.3664	0.3955	0.4463	0.4921	0.4992
r_Ω	0.7368	0.7375	0.8041	1	–	–



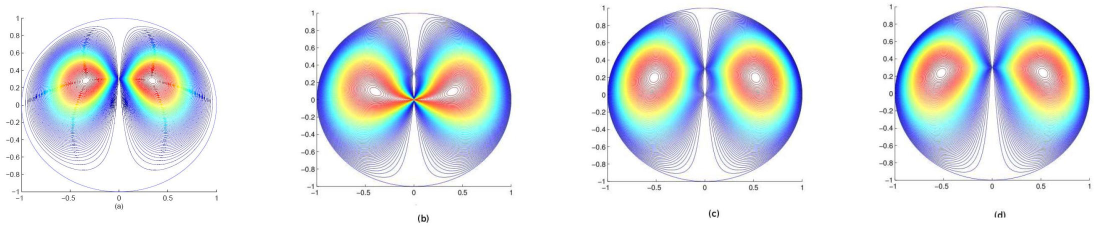
2. **An axisymmetric Stokeslet of strength** $(0, 0, \frac{F}{8\pi})$ **located at** $(0, 0, c)$**,** $c < 1$: The perturbed flow inside $r = 1$ is

$$
\begin{aligned}
\psi(r,\theta) \quad &= \quad \frac{F}{8\pi}\sum_{n=1}^{\infty}\left[\frac{A_n}{r^n} + \frac{B_n}{r^{n-2}} - \left(\frac{A_n(2n+3) + B_n(1+2\lambda)(2n+1)}{2(1+\lambda(2n+1))}\right) r^{n+1}\right. \\
&\left. + \left(\frac{A_n(1-2\lambda)(2n+1) + B_n(2n-1)}{2(1+\lambda(2n+1))}\right) r^{n+3}\right] P_n^1(\cos\theta)\sin\theta, \quad A_n = -\frac{c^{n+1}}{(2n+3)}, B_n = \frac{c^{n-1}}{(2n-1)}.
\end{aligned}
$$

Locations of the points of zero-vorticity for $\lambda = 0.5, 1, 10, 100$ for a Stokeslet located at $(0, 0, 0.3)$ are as follows.

θ	0.8	1.0	1.8	2.0	2.6	3.0
r_Ω	0.4484	0.3952	0.3956	0.3588	0.5212	0.5716

The only stagnation points were found to be at $(0, 0, 1)$ and $(0, 0, -1)$. Some flow patterns for the off-centered Stokeslet flow in a container (plotted using our exact solution) are shown below for several values of λ ((a) $\lambda = 0$, (b) $\lambda = 0.2$, (c) $\lambda = 0.3$, (d) $\lambda > 0.5$). The eddy pattern exists for all values of λ due to the presence of separation in the flow domain.



CONCLUSIONS

We studied the problem of a Stokes flow generated due to a Stokeslet inside a spherical cavity using Navier slip boundary conditions. We observed that the slip parameter has an effect on the flow patterns in the axisymmetric flows and that the stagnation points and points of zero vorticity change with the slip parameter λ. In the case of an axisymmetric Stokes flow due to a Stokeslet at the origin [3], we observed that the eddy patterns vanish for $\lambda \geq 0.5$. We conjectured that the co-existence of both the separation points and points of zero vorticity in the flow domain leads to the formation of eddies for $0 \leq \lambda < 0.5$. We showed that for an axisymmetric off-centred Stokeslet, there are only two stagnation points and that the set of points of zero vorticity do not change for $\lambda \geq 0.5$. In this case the flow patterns exist for all values of λ. We conjecture that the eddy patterns do not change for $\lambda \geq 0.5$ due to the existence of the same set of points of zero vorticity in this case. We observed that the phenomena of Stokeslet reversal [4] exists only in the axisymmetric off-centred Stokeslet case, where it is evident that it is dependent on the slip parameter.

References

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