SLIPPING STOKES FLOW ABOUT A SOLID PARTICLE

Antoine Sellier
LadHyx, Ecole polytechnique, 91128 Palaiseau Cédex, France

Summary We examine the settling rigid-body migration of a solid and slip particle immersed in a Newtonian liquid and subject to a uniform gravity field. The Navier slip condition is required at the particle surface and a new boundary approach is establish to accurately compute the settling motion of arbitrarily-shaped slip particles. Benchmark tests against analytical solutions for sphere and spheroids and new results for a fully-three-dimensional ellipsoidal particle will be presented.

PROBLEM AND MOTIVATING ISSUES

Governing problem

We consider, as sketched in Figure 1, a solid and slip particle $\mathcal{P}$ immersed in an unbounded Newtonian liquid with uniform density ρ and viscosity μ. Both the particle and the fluid are subject to a uniform gravity field $\mathbf{g}$ and the particle has uniform density ρ_s, center of mass O and smooth surface S with unit normal pointing into the liquid. It furthermore experiences a solid-body migration having angular velocity $\mathbf{\Omega}$ and translational velocity $\mathbf{U}$ with $\mathbf{U}$ the velocity of the center of mass O.

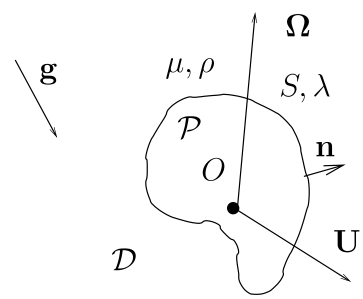

Figure 1. A solid and slip particle $\mathcal{P}$ experiencing a rigid-body migration with translational velocity $\mathbf{U}$ (the velocity of its center of mass O) and angular velocity $\mathbf{\Omega}$.

The settling particle has typical length scale a and, setting $\mathbf{x} = \mathbf{OM}$, the flow about it has pressure field $p + \rho\mathbf{g}.\mathbf{x}$ and velocity field $\mathbf{u}$ with typical magnitude V. This flow $(\mathbf{u}, p)$ is quiescent far from the particle and assuming that $\mathrm{Re} = \rho V a/\mu$ vanishes, i. e. negligible inertial effects, $(\mathbf{u}, p)$ then obeys in the liquid domain $\mathcal{D}$ the following steady Stokes equations and far-field behaviour

$$\mu\nabla^2\mathbf{u} = \nabla p \;\text{ and }\; \nabla.\mathbf{u} = 0 \;\text{ in }\; \mathcal{D}, (\mathbf{u}, p) \to (\mathbf{0}, 0) \quad \text{as } |\mathbf{x}| \to \infty. \tag{1}$$

At the smooth and slip particle surface S the flow $(\mathbf{u}, p)$ with stress tensor $\boldsymbol{\sigma}$ is allowed to slip. This is here modelized by enforcing the widely-employed Navier (1823) condition

$$\mathbf{u}(M) = \mathbf{U} + \mathbf{\Omega} \wedge \mathbf{OM} + \lambda\{\boldsymbol{\sigma}.\mathbf{n} - (\mathbf{n}.\boldsymbol{\sigma}.\mathbf{n})\boldsymbol{\sigma}.\mathbf{n}\} \;\text{ on } S \tag{2}$$

with $\lambda \geq 0$ the so-called lip length and $\mathbf{n}$ the unit normal directed into the liquid. The flow $(\mathbf{u}, p)$ with stress tensor $\boldsymbol{\sigma}$ exerts on the particle surface the traction $\mathbf{f} = \boldsymbol{\sigma}.\mathbf{n}$. Neglecting the particle inertia and therefore requiring the settling particle to be force-free and torque-free, one finally supplements (1)-(2) with the additional relations (recall that O is the particle center of mass)

$$\int_S \boldsymbol{\sigma}.\mathbf{n} dS = (\rho - \rho_s)\mathcal{V}\mathbf{g}, \qquad \int_S \mathbf{x} \wedge \boldsymbol{\sigma}.\mathbf{n} dS = \mathbf{0} \tag{3}$$

where $\mathcal{V}$ designates the particle volume.

Challenging issues and motivations

Solving (1)-(3), i. e. determining the settling motion $(\mathbf{U}, \mathbf{\Omega})$ and the flow $(\mathbf{u}, p)$, for a given particle and settings $(\lambda, \mathbf{g})$ turns out to be a very very challenging task as soon as the particle is non-spherical. This explains why the available literature is restricted to a sphere (see Basset 1961) and to *axisymmetric neary-spherical* (Senchenko and Keh 2006; Chang and Keh 2009) or *spheroidal* (Keh and Huang 2004; Keh and Chang 2008) particles translating along the axis of revolution. To the author's very best knowledge a treatment valid for arbitrarily-shaped particles has not yet been proposed. This work thus presents a new procedure to accurately determine at a reasonable cpu-time cost the settling motion $(\mathbf{U}, \mathbf{\Omega})$ for a solid and slipping particle with arbitrary shape and slip-length parameter λ.

Corresponding author. E-mail: sellier@ladhyx.polytechnique.fr.

ADVOCATED NEW BOUNDARY APPROACH

Relevant boundary-integral equation

Let us resort to Cartesian coordinates (O, x_1, x_2, x_3) and prescribe the particle rigid-body motion $(\mathbf{U}, \mathbf{\Omega})$ therefore ending up with a Stokes flow $(\mathbf{u}, p)$ about the particle obeying (1)-(2). This flow exerts on the particle surface S a traction $\mathbf{f}$ and we first establish that the density $\mathbf{f}$ is governed by the following boundary-integral equation

$$[\mathbf{U} + \mathbf{x} \wedge \mathbf{\Omega}].\mathbf{e}_j = -\lambda[\mathbf{f} - (\mathbf{f}.\mathbf{n})\mathbf{n}](\mathbf{x}).\mathbf{e}_j - \frac{1}{8\pi\mu}\int_S G_{ij}(\mathbf{y}, \mathbf{x})[\mathbf{f}(\mathbf{y}).\mathbf{e}_i dS$$
$$+\frac{1}{8\pi}\int_S \{[\mathbf{f} - (\mathbf{f}.\mathbf{n})\mathbf{n}](\mathbf{y}) - [\mathbf{f} - (\mathbf{f}.\mathbf{n})\mathbf{n}](\mathbf{x})\}.\mathbf{e}_i T_{ijk}(\mathbf{y}, \mathbf{x})\mathbf{n}(\mathbf{y}).\mathbf{e}_k dS \text{ for } j = 1, 2, 3 \text{ and } \mathbf{x} \text{ on } S. \quad (4)$$

For convenience, we shall further set $\mathbf{u} = \mathbf{u}_t^{(i)}$ and $\mathbf{f} = \mathbf{f}_t^{(i)}$ when $(\mathbf{U}, \mathbf{\Omega}) = (\mathbf{e}_i, \mathbf{0})$ and also $\mathbf{u} = \mathbf{u}_r^{(i)}$ and $\mathbf{f} = \mathbf{f}_r^{(i)}$ when $(\mathbf{U}, \mathbf{\Omega}) = (\mathbf{0}, \mathbf{e}_i)$.

A key linear system

Exploiting the reciprocal identity (Pozrikidis) for the Stokes flow solution to the problem (1)-(3) and the above flows $\mathbf{u}_t^{(i)}$ and $\mathbf{u}_r^{(i)}$ then makes it possible to cast the relations (3) into a linear system for the unknown particle settling velocities $\mathbf{U}$ and $\mathbf{\Omega}$. Setting $U_j = \mathbf{U}.\mathbf{e}_j$ and $\Omega_j = \mathbf{\Omega}.\mathbf{e}_j$ and curtailing the details one actually arrives at the key equations

$$\{\int_S [\mathbf{u}_t^{(i)} - \mathbf{e}_i + \lambda(\mathbf{f}_t^{(i)}.\mathbf{n})\mathbf{n}].\mathbf{e}_j dS\}U_j + \{\int_S [\mathbf{u}_t^{(i)} - \mathbf{e}_i + \lambda(\mathbf{f}_t^{(i)}.\mathbf{n})\mathbf{n}].(\mathbf{e}_j \wedge \mathbf{x})dS\}\Omega_j = \lambda(\rho - \rho_s)\mathcal{V}\mathbf{g}.\mathbf{e}_i \quad (5)$$
$$\{\int_S [\mathbf{u}_r^{(i)} - \mathbf{e}_i \wedge \mathbf{x} + \lambda(\mathbf{f}_r^{(i)}.\mathbf{n})\mathbf{n}].\mathbf{e}_j dS\}U_j + \{\int_S [\mathbf{u}_r^{(i)} - \mathbf{e}_i \wedge \mathbf{x} + \lambda(\mathbf{f}_r^{(i)}.\mathbf{n})\mathbf{n}].(\mathbf{e}_j \wedge \mathbf{x})dS\}\Omega_j = 0 \quad (6)$$

where in (5)-(6) there is a summation over the repeated indices j, k and $i = 1, 2, 3$. Upon determining on the particle surface the occurrig vectors $\mathbf{u}_t^{(i)}, \mathbf{f}_t^{(i)}$ and $\mathbf{u}_r^{(i)}, \mathbf{f}_r^{(i)}$, one then gains the velocities $\mathbf{U}$ and $\mathbf{\Omega}$ from the linear system (5)-(6).

PROPOSED PRESENTATION

The presentation will successively address the following issues:

1) First, establish the proposed boundary-integral equation (4) and the linear system (5)-(6).

2) Propose and implement a suitable numerical strategy to accurately invert the encountered integral equations and linear system and therefore numerically determine the particle incurred rigid-body settling $(\mathbf{U}, \mathbf{\Omega})$.

3) Benchmark the advocated procedure against the analytical results obtained by Basset (1961) for a rotating or translating sphere and Basset and the analytical results obtained by Keh & Huang (2004) and Keh & Chang (2008) for a spheroid (either oblate or prolate) translating parallel with its axis of revolution.

4) Present and discuss new results for a non-spheroidal (and therefore fully three-dimensional) ellipsoid.

References

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