RECIPROCITY AMONG ELEMENTAL RELAXATION AND DRIVEN-FLOW PROBLEMS FOR A RAREFIED GAS

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Summary We consider a rarefied gas between two parallel plates and discuss the reciprocity among elemental relaxation and driven-flow problems described by a linearized kinetic equation. Numerical demonstration of the reciprocity based on the Bhatnagar-Gross-Krook model is presented . Moreover, a propagation of an edge, as well as a discontinuity, of the velocity distribution function is demonstrated.

PROBLEM

Consider a rarefied gas between two parallel plates separated by a distance D. We denote a position vector by $D\boldsymbol{x}$ and introduce the rectangular coordinates so that the two plates are located at $x_1 = \pm 1/2$. For a simple demonstration, we assume that (i) the gas behavior is described by the Bhatnagar–Gross–Krook (BGK) model [1] kinetic equation and that (ii) the gas molecules are diffusely reflected on the plates. We investigate the gas behavior in the following four cases where the deviation from the reference resting equilibrium state with density ρ_0 and temperature T_0 (or pressure $p_0 = \rho_0 RT_0$, where R is the specific gas constant) is so small that the linearization of the equation and boundary condition is allowed. In what follows, the time is denoted by $t_0 t$, the molecular velocity by $(2RT_0)^{1/2}\boldsymbol{\zeta}$, the velocity distribution function (VDF) by $\rho_0(2RT_0)^{-3/2}(1+\psi)E$ with $E = \pi^{-3/2}\exp(-|\boldsymbol{\zeta}|^2)$, where $t_0 = D/(2RT_0)^{1/2}$ is a reference time.

(DP) A time evolution from $\psi = 0$ when the plates are kept at the temperature T_0 and a uniform external force $(0, -F, 0)$ such that the ratio $C_{DP} = FD/mRT_0$ (m is the mass of a molecule) is small, is acting on the gas.

(DT) A time evolution from $\psi = C_{DT}x_2(|\boldsymbol{\zeta}|^2 - 5/2)$ when the plates are kept at temperature $T_0(1 + C_{DT}x_2)$ ($|C_{DT}| \ll 1$).

(RP) A time relaxation from $\psi = -C_{RP}\zeta_2$ [i.e., from a uniform flow field $(0, -U, 0)$, where $C_{RP} = 2U/(2RT_0)^{1/2}$ and $|C_{RP}| \ll 1$] when the plates are kept at the temperature T_0.

(RT) A time relaxation from $\psi = -C_{RT}\zeta_2(|\boldsymbol{\zeta}|^2 - 5/2)$ [i.e., from a uniform heat flow $(0, -q, 0)$, where $C_{RT} = 2q/p_0(2RT_0)^{1/2}$ and $|C_{RT}| \ll 1$] when the plates are kept at the temperature T_0.

We shall use below the subscript index $\alpha = DP, DT, RP, RT$ to identify the problem, for instance, ψ_{DP}, ψ_{RT}, etc.

BACKGROUND AND PRIMARY MOTIVATION: RECIPROCITY OF FLUXES

Extensions of the Onsager reciprocity for rarefied gas flows has been discussed by several authors [2] in a connection to the entropy production. The first author has recently proposed an alternative way of the reciprocity argument and has developed, for the first time, a general framework that covers not only time-independent but also time-dependent problems [3, 4] for an arbitrary Knudsen number. We have chosen the above elemental problems for its simple demonstration.
Let us denote the flow velocity by $(2RT_0)^{1/2}C_\alpha(0, u_\alpha, 0)$, and the heat flow vector by $p_0(2RT_0)^{1/2}C_\alpha(0, Q_\alpha, 0)$, where α is the problem index. Application of the general theory [4] leads to the following identities that hold at any time t (≥ 0) for an arbitrary Knudsen number:

$$\int_0^t M_{RT}dt = H_{DP} = M_{DT} = \int_0^t H_{RP}dt, \quad M_{DP} = \int_0^t M_{RP}dt, \quad H_{DT} = \int_0^t H_{RT}dt, \tag{1}$$

where $(M_\alpha, H_\alpha) = \int_{-\frac{1}{2}}^{\frac{1}{2}}(u_\alpha, Q_\alpha)dx_1$. Below we demonstrate these newly found identities numerically.

NUMERICAL DEMONSTRATION USING THE BGK MODEL

Flux reciprocity

As an example, Figure 1 shows the profiles of the dimensionless mass and heat flow, u_α's and Q_α's, in the case of $k = 1$, where k is the Knudsen number multiplied by $\sqrt{\pi}/2$. They are all different from each other, as seen from the figure. Nevertheless, it is clearly seen from Figure 2 that the equalities in (1) hold *at any time* for a wide range of the Knudsen number (or k).

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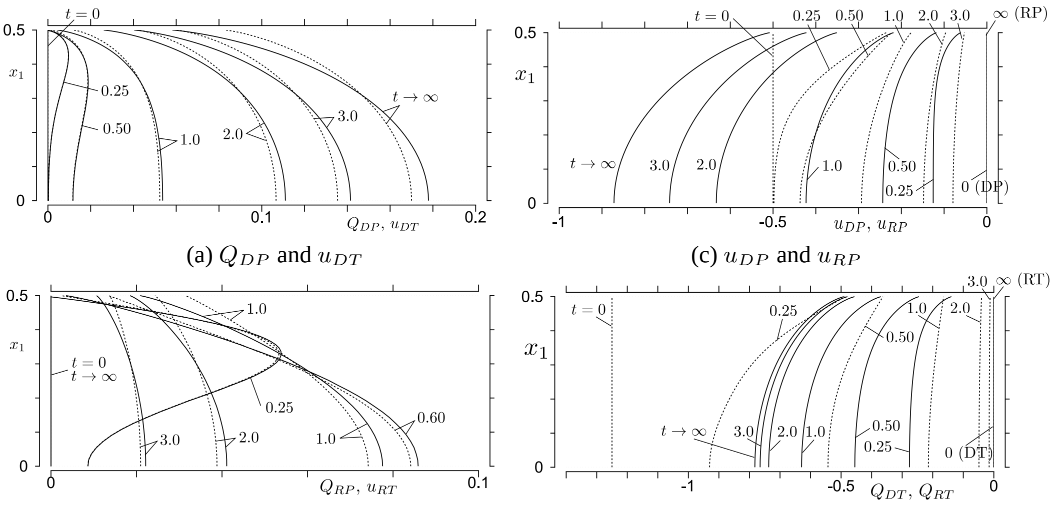
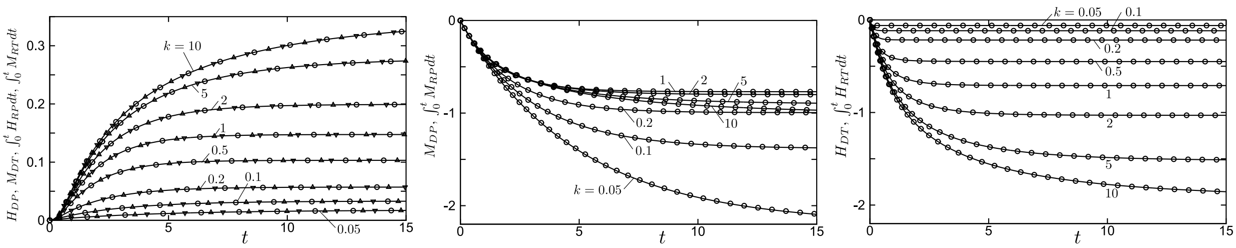


(a) Q_{DP} and u_{DT} (c) u_{DP} and u_{RP}

(b) Q_{RP} and u_{RT} (d) Q_{DT} and Q_{RT}

Figure 1: Mass and heat flows in a half of the channel $0 \le x_1 \le 1/2$ for $k = 1$. (a) u_{DT} and Q_{DP}, (b) u_{RT} and Q_{RP}, (c) u_{DP} and u_{RP}, and (d) Q_{DT} and Q_{RT}. Solid lines: Q_{DP}, Q_{RP}, u_{DP}, and Q_{DT}. Dashed lines: u_{DT}, u_{RT}, u_{RP}, and Q_{RT}.



(a) First equation in (1) (b) Second equation in (1) (c) Third equation in (1)

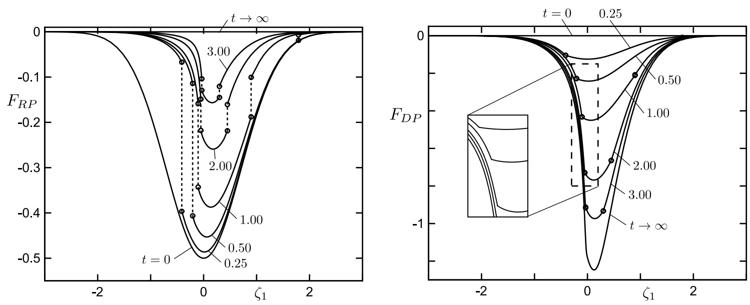
Figure 2: Time evolution of the fluxes in Eq. (1) for various k. Solid lines: H_{DP} in (a), M_{DP} in (b), and H_{DT} in (c). Circles: M_{DT} in (a), $\int_0^t M_{RP}dt$ in (b), and $\int_0^t H_{RT}dt$ in (c). Triangles in (a): $\int_0^t H_{RP}dt$. Inverted triangles in (a): $\int_0^t M_{RT}dt$.

Propagation of the discontinuities of the VDF and its derivative

Figure 3 shows an example of the marginal VDFs defined by

$$F_\alpha = \sqrt{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_\alpha E d\zeta_2 d\zeta_3.$$

It is seen from the figure that F_{RP} has discontinuities, while F_{DP} does not. The discontinuities of the former come from the fact that the initial and boundary data are not continuously connected to each other, and the discontinuities on the boundaries at the initial time propagates into a gas region along the free-flight molecular trajectories. This kind of source at the initial time does not exist for the problem (RP), so that F_{DP} is continuous. On the other hand, a close observation shows that its derivative, namely $\partial F_{DP}/\partial \zeta_1$, has discontinuities. This discontinuity can be understood in the connection to that in (RP), because we can show by a simple argument that a more detailed identity $\psi_{DP} = \int_0^t \psi_{RP} dt$ does hold. A similar correspondence can also be obtained between ψ_{DT} and ψ_{RT}. Our demonstration thus clearly shows the occurrence of an edge of VDF and its propagation, as well as the known discontinuities.



(a) F_{RP} (b) F_{DP}

Figure 3: Marginal VDF of problems (RP) and (DP) at $x_1 = 0.397$ for $k = 1$.

References

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